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ROBERT E. PETERSON'S CHEAP EDUCATIONAL SERIES.

A

TREATISE ON ALGEBRA,

CONTAINING THE

MOST USEFUL PARTS OF THAT SCIENCE,

ILLUSTRATED BY

A COPIOUS COLLECTION OF EXAMPLES,

DESIGNED FOR THE USE OF SCHOOLS AND COLLEGES.

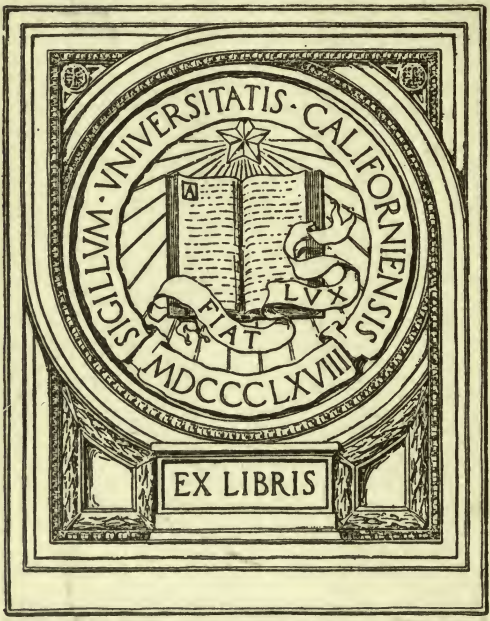
BY ENOCH LEWIS.

FOURTH EDITION, REVISED AND ENLARGED.

PHILADELPHIA:
PUBLISHED BY ROBERT E. PETERSON,
S. W. CORNER OF FIFTH AND ARCH STREETS.

1852.

IN MEMORIAM
FLORIAN CAJORI



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PREFACE TO THE FOURTH EDITION.

Since the first edition of this work was issued, a number of years have elapsed, during which several treatises on this branch of science have been given to the public; some of which are evidently the productions of able hands; still the reasons which induced the author of The Practical Analyst to offer his work to public acceptance, appear applicable to the present day. The generality of pupils who undertake to acquire a knowledge of algebra, require a manual which includes the general principles, and processes of the science without needless expansion, or very abstruse speculations. The work has been carefully revised, and such changes introduced as were judged needful to adapt it to the present improved state of the science. A number of examples, not contained in the former editions, have been inserted in this, and the original object of smoothing the path of the pupil and diminishing the labor of the instructor, has been kept steadily in view. Though the design of this treatise is rather to meet the wants of students in our schools, than to supply that class who may incline to dive into the most refined intricacies of the analytic art; yet, even to the latter class, this tract will probably be found a safe and correct introduction.

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PREFACE.

The mathematical sciences have been considered, since the early periods of their existence, among the noblest objects of human inquiry. Of these, next to arithmetic, geometry occupies the first place, in regard both to importance and to time. The Grecian philosophers cultivated this branch of knowledge, with an ardor and industry, which manifest their high opinion of its value. In the Ionian and Pythagorean schools, geometry was considered an indispensable preliminary to the study of philosophy. When a person, ignorant of geometry, applied for his instructions to the philosopher Xenocrates, he is said to have made this laconic reply: "Thou hast not the handles of philosophy."

The perspicuity of geometrical reasoning, the accurate and inimitable dependence of its arguments, and the unfaltering certainty of its conclusions, are eminently calculated to form the mind to habits of attention, and to a regular and forcible concatenation of its ideas. No wonder, then, that it was so highly prized by that acute and inquisitive people.

Two modes of procedure, according to the different objects in view, were adopted by the ancient geometers: synthesis and analysis.

Synthesis, or composition, consists in the direct solution of a problem; or, the demonstration of a proposition by a series of arguments, regularly deduced, from self-evident truths, or from other propositions previously established. This method is very proper for conveying, with clearness, to another, those truths which are completely understood by the instructor. The works of the ancient geometers, which have escaped the ravages of the barbarous ages, are mostly synthetical. The Elements of Euclid, so well

known in our modern schools of geometry, furnish, probably, the most complete specimen of the ancient synthesis extant. And it is a remarkable fact in the history of science, that the geometry of Euclid, though written two thousand years ago, and some time anterior to any other mathematical tract that has reached us, is still one of the best elementary works on that subject which has ever appeared.

Analysis, or decomposition, on the other hand, assumes, as known, the proposition which is to be examined; or, as already effected, the solution which is to be made, and thence proceeds to examine the consequences necessarily resulting from such supposition, until, in case of a theorem, a conclusion is attained, the truth or falsehood of which is already known, whence the correctness or absurdity of the supposition becomes established; or, in case of a problem, such relations are determined as prove the possibility or impossibility of the solution. In synthesis, observes Montucla, we proceed from the known to the unknown, from the trunk to the branches; in analysis, we proceed from the unknown to the known, from the branches to the trunk.

The analytical method is frequently indispensable, when new problems are to be solved, or new theorems investigated. No doubt, many propositions which the ancients have transmitted to us in the synthetical form, owe their discoveries to analysis.

Of this method of procedure, the mathematical collections of Pappus, and the work, *De Sectiones Rationis*, of Apollonius, furnish the principal specimens which the ancient geometers have left us.

In the ancient geometry, the magnitudes under consideration, were mostly presented to the mind through the medium of representations, as similar as practicable to their antitypes. In some instances, however, this analogy was entirely abandoned, as in the fifth book of Euclid's Elements, where right lines are the only representatives used, yet the reasoning is so conducted, as to be equally applicable to magnitudes of every kind, and even to abstract num-

bers. In this instance, we may discover a commencement of that species of generalization, which forms so conspicuous a feature in our modern mathematics.

The ancient analysis furnished the germ of that branch of mathematics, which, in the hands of the moderns, has become the great master key to all the rest; but it does not appear to have assumed the character of a distinct science, till after the commencement of the Christian era.

The earliest writer, in whose works the science of algebra is distinctly seen, was Diophantus, a mathematician of the Alexandrian school. The time in which he lived is not precisely known, but it was not later than the fourth century, as the daughter of Theon, the amiable and accomplished Hypathia, who died about the beginning of the fifth, wrote a commentary on his works.

Whether Diophantus was the inventor, or only an improver of algebra, cannot now be known; the latter supposition, however, is the more probable, as the science in his hands exhibits a degree of maturity, which it can hardly be supposed to have attained in the first period of its existence.

A part only of the original work of Diophantus is now to be found; in this he does not explain the first principles of the science, but teaches the solution of a great variety of difficult questions, in that branch of the subject, now called, from him, the Diophantine Algebra, or the *indeterminate analysis*, applied to equations of the higher orders. The various questions, if original, which he has formed, and the address with which he has conducted their solutions, necessarily inspire his readers with a high opinion of his invention and discernment. His work, written in the original Greek, was discovered in the Vatican Library, about the middle of the sixteenth century.

Though the science of algebra appears to have originated among the Greeks of the Alexandrian school, the inhabitants of western Europe derived their knowledge of it from the Arabs, who are by some supposed to have been its inventors.

Dr. Wallis observes, that they differ essentially from

Diophantus in their manner of expressing the powers. The Greek analyst calls the 2d, 3d, 4th, 5th, 6th, etc. powers, the square, cube, squared square, squared cube, cubed cube, etc.; each power being designated by the two inferior powers of which it is the product. But the Arabian algebraists denominate the 5th power the first sursolid, the 6th the squared cube; being the square of the cube, and not the product of a square and cube; the 7th the second sursolid, and so of the other powers. Hence, he infers, that the Greek and Arabian analyses were not derived from a common source.

With due regard for the opinion of this eminent scholar, it appears quite as rational to suppose, that the Arabian mathematicians may have borrowed from their tutors, the Greeks, this branch, with the rest of the mutilated sciences, and adopted, in the denomination of their powers, a mode of expression peculiar to themselves, as to believe, that while the other parts of Grecian science were sought with avidity, this was permitted to sleep amidst the dust of neglected libraries, till the same thing had been reinvented by a people much less advanced in scientific knowledge, and less remarkable for invention than their predecessors.*

Whether the science of Algebra was *invented* by the Arabs, or borrowed from the Greeks, the name is unquestionably of Arabic origin. The names given by the Arabs, for they used a plurality, were *algebra valmucabala.* These words, according to Lucas de Burgo, signify *restauratio et oppositio*, restoration or rebuilding, and opposition. Golius defines the word gebera, or giabera, by *religavit consolidavit*, it bound or consolidated; and mocabulat, by *comparatio oppositio*, comparison, opposition.

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* In the progress of science, there always appears a regular dependence in the successive stages. The discoveries of one age are the results of those made in the former. Logarithms were invented, when the discoveries in astronomy and trigonometry had rendered their use indispensable. The discoveries of Newton could not have been made, even by that gigantic genius, in the time of Copernicus; nor could Columbus have led his trembling companions across the Atlantic, before the invention of the mariner's compass.

By these words they probably designed to indicate the general objects of the science. The quantity whose value is sought, is commonly interwoven with, or *bound* to other quantities, in such a manner as to form one or more equations or *comparisons* of quantities set in *opposition* to each other. These equations are then transformed, or *rebuilt*, till the unknown quantity is brought out in *opposition* to a given or known one.*

The most ancient authors on algebra among the Arabs, are Mohammed ben Musa, and Thebit ben Corah. The former is described by Cardan as the inventor of the method of solving equations of the second degree, a discovery in which he was certainly anticipated by the mathematicians of the Alexandrian school. From the title given to his book, it has been inferred, that he flourished during the reign of Almamon, or in the early part of the ninth century.

Whether the Arabian algebraists proceeded beyond the solution of equations of the second degree, is an unsettled question. An accurate knowledge of the mathematical sciences is seldom combined, in the same individual, with an extensive acquaintance with Arabic literature, and, therefore, little is certainly known on this subject. The Bodleian Library in England, and that of the Escurial in Spain, are said to possess a great number of Arabic works on the subject of algebra.

Leonard of Pisa, who lived near the beginning of the thirteenth century, impelled by a thirst for mathematical knowledge, traveled into Arabia and other parts of the east, and on his return, first communicated to his countrymen the science of algebra. I do not find that any of his writings on that subject have ever been published.

In a treatise upon trigonometry, by Regiomontanus, of Franconia, written about the year 1464, some problems are solved by algebra, in which he refers to the rules, as though commonly known.

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* The name *almucabala* was adopted by some of the Italian writers, and it is thus designated in some of the works of Cardan; but the term algebra appears now irrevocably fixed upon it.

But the earliest European author, whose works, specially on this subject, have been published, was Lucas de Burgo, before mentioned. He was a Franciscan, who traveled in the east, either in pursuit of knowledge, or for some purpose not now known, and after his return, taught mathematics at Naples, Venice, and Milan. His work, in which the rules of algebra are laid down, was first printed in 1494. In this the science appears very far below our modern algebra. The rules for the solution of adfected quadratic equations, are given in semi-barbarous Latin verse, and the different cases separately treated. His solutions do not rise to equations above the second degree.

The solution of cubic equations appears to have been first effected by Scipio Ferrei, professor of mathematics at Bologna, about the beginning of the sixteenth century.

This solution, however, included but one case, namely, that in which the first and third powers only, of the unknown quantity, were involved. Some questions, including this case, being afterward proposed to Nicholas Tartaglia, an eminent mathematician of Brescia, he discovered a general solution of equations of the third degree. This discovery he communicated to Jerome Cardan, a physician of Milan, under an injunction of secrecy. Cardan, notwithstanding, having found the demonstration, published it in his work, De Arte Magna, in 1545. This is now commonly called the method of Cardan.

Cardan first remarked the plurality of roots in quadratic equations, and their distinction into positive and negative. The solution of equations of the fourth degree was accomplished soon after the discovery of Tartaglia, by Lewis Ferrari, a pupil and coadjutor of Cardan.

Some intricacies belonging to equations of the third degree were further unraveled by Raphael Bombelli, of Bologna, and published in 1579.

The algebraists, whose labors have been noted, expressed known quantities by their proper numerical characters, and therefore their solutions were destitute of that generality which constitutes so prominent a feature of our present analysis. Their *modes* of solution were applicable to all

similar problems, but their *results* were confined to particular questions. Francis Vieta,* by using letters of the alphabet to represent known as well as unknown quantities, gave an extent and generality to the science which it did not possess before. By this change, algebraists were enabled to include, in a single solution, a whole class of problems, and to obtain the result in each particular case, by simple substitution. He taught the method of taking, from an equation, its second term, and thus reducing affected quadratics to simple quadratics; and all cases of cubic equations to the case solved by Ferrei. He also taught the solution of cubic equations, having three possible roots, by the trisection of an angle. He made numerous improvements in algebra, and furnished the germs of some discoveries which have since grown up under other names.

Thomas Harriott, an English analyst, followed in the steps of Vieta, and made several important improvements in the science. He first adopted the plan of placing all the terms of an equation on one side of the sign of equality, and zero on the other; and showed that an equation thus expressed, may be always formed by the multiplication of binomial factors. This naturally led to the discovery, that every equation has as many roots, or values of the unknown quantity, as there are units in the index of its highest power. Harriott, notwithstanding this observation, lay directly in his road, does not appear to have made it. The complete development of negative and impossible roots, was left to exercise the ingenuity of succeeding inquirers. Vieta employed the large letters of the alphabet, and indicated their powers by initials placed over them, as exponents are now used; Harriott substituted small letters, and denoted their powers by repetitions of the letter; thus instead of A^q, A^c, etc., he wrote *aa*, *aaa*, etc., a small change, indeed, but still an obvious improvement in the notation. His Artis Analyticæ Praxis, containing his dis-

* This eminent mathematician was born at Fontenoy, in Poitou, in 1540, and died in 1603.

coveries, was published after his death.* He was born at Oxford, in 1560, and died in 1621.

The celebrated French philosopher, René Descartes, contributed largely to the advancement of algebra. He explained the nature of *negative* roots, and taught the manner of finding their number by the changes of the signs in the general equation. The use of exponents, as now applied, is attributed to him; as is also the method of indeterminate co-efficients.

The first application of algebra to geometry, was long prior to the time of Descartes, yet those sciences are indebted to him for that intimate union, which has since contributed so extensively to the improvement of both. Descartes was born in 1596, and died in 1650.

Soon after the time of Descartes, the analytical science took a flight, which, if followed, would lead me far beyond the bounds of a preface. This sketch of the history will, therefore, be closed with the remark, that this science, as enriched by the discoveries of Newton, Leibnitz, and others, has become, in the hands of our modern philosophers, the torch to guide them through the most intricate labyrinths of science; that by its light they have traced the motions of the celestial bodies through all their mystic dance, and penetrated many of the recesses of nature, where, without its aid, they must have been bewildered and lost.

The following work was undertaken from a persuasion, that the books on algebra, used in our schools, were none of them entirely adapted to the wants of a large class of pupils, many of whom do not enjoy the leisure, or the talents requisite for penetrating the depths of science, and yet are desirous of attaining a knowledge of this subject, sufficient to qualify them for studying successfully the common practical branches of mathematics, to which this serves as a key. In the most popular treatise on this science, with which our schools have been furnished, the progress of the student appears to me needlessly obstructed, by diffi-

* Bossut says it was published in 1620; Montucla and others say in 1631.

culties near the commencement, which, to a common intellect, are almost insuperable.

My object has been to present the most useful parts of the science in such order, that no very abstruse process should be required, before the pupil had been sufficiently exercised, to acquire the requisite skill. The expedients demanded for solving the questions are mostly pointed out before they are called for in practice.

In a popular treatise on a subject which has engaged the attention of so great a number of authors, many of them, unquestionably, endued with talents of the highest order, it would be idle to expect much originality. My object has been to smooth the path of the student, and diminish the toil of the tutor. The work, with all its imperfections, is submitted to the inspection of the public.

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ALGEBRA.

DEFINITIONS.

ARTICLE 1. Algebra, or specious arithmetic, is the science of computing by symbols or general characters.

2. Quantities, of whatever kind, are usually denoted by letters of the alphabet.

3. The relations of quantities, and the operations to be performed on them, are indicated by the following characters, thus:

4. The sign + *plus*, or *more*, indicates addition, as $a+b$ signifies that b is added to a.

5. The symbol — *minus*, or *less*, indicates subtraction; thus, $a—b$, signifies that b is subtracted from a.

The characters + and — are called, by way of eminence, the *signs* of the quantities to which they are prefixed.

6. Multiplication is denoted by the sign $\times$, *into*, placed between the factors, as $a \times b$; or by a period, as $a.b$; or, more frequently, by joining the letters, like letters in a word, as ab; each of which expressions denotes the product of a and b.

7. Division is indicated by the sign $\div$ *by*, placed between the terms; or by writing the dividend above, and the divisor below, a horizontal line; thus $a \div b$, or $\frac{a}{b}$ denotes the quotient of a when divided by b.

8. The difference of two quantities, when it is unknown which is the greater, is indicated by the sign $\sim$, placed between them; thus, $a \sim b$, or $b \sim a$, denotes the difference of a and b.

9. : : : : indicates *proportion;* thus, $a : b :: c : d$, may be read, a has to b the same ratio that c has to d.

10. $=$ *equal to*, signifies that the quantities between which it is placed, are equal to each other; thus, $b+c = a-d+e$, is a combination, called an *equation*, which signifies, that when the operations indicated by the signs are performed on the quantities placed on each side of the sign of equality, the results are equal to each other.

11. A *simple quantity* is that which consists of one term only, viz. a quantity denoted by a single letter, or several letters and figures, connected by the sign of multiplication or division, expressed or understood,* as a, abc, $5cd$; $\frac{cb}{d}$.

12. A *compound quantity* consists of two or more simple quantities connected by the signs of addition or subtraction, as $a-b$, $ab-ac+bd$.

13. A *root* is a number or quantity, from which a power is conceived to arise.

14. A *power* of a number or root, is the product of a unit, multiplied continually by the given root, any proposed number of times; and the figure or quantity which indicates the number of multiplications thus made, is called the *index* or *exponent;* thus, $1 \times 5 \times 5 \times 5 \times 5 = 625$; and, $1 \times a \times a \times a \times a \times a$, or $aaaaa$, are the 4th and 5th powers of 5 and a respectively, and are usually expressed by the root with the index of the power set over it: as 5^4, a^5; hence $a^0 = 1$, whatever value may be assigned to a.

15. The second power is called the *square;* the third power, the *cube;* the fourth power, the *biquadrate*, etc. of their respective roots.

* An absolute number, though containing numerous digits, is considered as a simple quantity.

16. The *radical sign** $\surd$, prefixed to a quantity, indicates the square root; $\sqrt[3]{}$, the cube root; $\sqrt[4]{}$, the fourth root, etc. Roots are also expressed by fractional exponents; thus, $\surd a$, or $a^{\frac{1}{2}}$, denotes the square root of a; $\sqrt[3]{a}$, or $a^{\frac{1}{3}}$, the cube root of a; and $\sqrt[3]{a^2}$, or $a^{\frac{2}{3}}$, the cube root of the square of a.

17. A root which cannot be accurately expressed in numbers, is called a *surd*, or *irrational* quantity, as $\surd 5$, $\sqrt[3]{5}$.

18. A quantity which has no radical sign, or which having a radical sign, admits of an accurate extraction of the root indicated by the sign, is called *rational*; thus a, $\surd 16$, $\sqrt[3]{b^3}$, are rational quantities.

19. When a compound quantity has a line drawn over it, or is enclosed in brackets, the operation indicated by a preceding or subsequent sign, is to be performed on the whole considered as a simple quantity; thus, $\overline{a+b-c\times d}$, and $(b+c)\times(d-e)$, signify that the compound quantities connected by the sign $\times$, are multiplied together; and $\surd\overline{ab+dc}$, $\sqrt[3]{\{ab-cd+ef,\}}$ signify the square root, and the cube root respectively, of the compound quantities which are preceded by the radical signs.

20. A number prefixed to a letter, or combination of letters, is called the *co-efficient*; thus, in the expression $3ab$, 3 is the co-efficient.

21. A compound quantity, consisting of two terms, is called a *binomial*, as $a+b$; one of three terms, a *trinomial*, as $ab+ac+de$.

22. *Like quantities* are those which consist of the same letters similarly involved; as ab, $3ab$, $5ab$.

23. *Unlike quantities* consist of different letters, or different powers of the same letters, as a, ab, $3a^2$, $4ab^3$.

24. A simple quantity is usually termed a monomial.

*So called from *radix*, a root.

A compound quantity is termed a binomial, a trinomial, or polynomial, according as it consists of two, three or many terms. Thus, $a+b$, $ab+cd$, are binomials, $a+b+c$, $ab+cd+ef$, are trinomials, etc.

In the solution of problems, it is usual to denote known or given quantities, by the initial letters, a, b, c, etc., and unknown ones, by the final letters, v, x, y, etc.

The following examples are given for the purpose of exercising the student in the application of the algebraic signs.

Required the numerical values of the following combinations, supposing $a=7$, $b=6$, $c=5$, $d=3$, $e=2$.

$a^3+4bc-3de=343+120-18=445.$

$(ab+cd)\times(3bc+4ad)=(42+15)\times(90+84)=57\times 174=9918.$

$a^2b+cd^2-3be=303.$

$b^3+a^2d-2ce-4bc=223.$

$(ac^3+2bd)\times 5ac=159425.$

$\frac{b^3}{6a+de^2}=4$ $\qquad$ $\sqrt{a^3+b^2+ad}=20$

$\frac{a^3b+13cd^3}{2a^2-3de}=47.6625$

$\frac{5ad^3-2ace^2}{7b+3d}=13\frac{2}{51}$

$\sqrt{4a^3+7d^3+53be}=13$

$\frac{a^3-a^2b+3ad^2-41ce}{ab-cd+ae}=\frac{68}{34}=2$

Section I.

ADDITION.

Case I.

25. *When the quantities are like, and have like signs.*

Add all the co-efficients together, and to their sum

annex the given literal expression, prefixing the common sign.*

EXAMPLES.

$3a$	$2be$	$2a^2b$	$-3xy^2$	$2a-3b$
$5a$	$3be$	$3a^2b$	$-xy^2$	$5a-6b$
$7a$	be	$4a^2b$	$-4xy^2$	$7a-5b$
$4a$	$5be$	a^2b	$-7xy^2$	$8a-4b$
$6a$	$8be$	$5a^2b$	$-9xy^2$	$9a-4b$
a	$9be$	a^2b	$-11xy^2$	$15a-b$
$26a$				

$5x^3y-7bc+ab$	$6xy^{\frac{1}{2}}-7ac$	$4ab^3+3a^2c-bc^3$
$3x^3y-6bc+4ab$	$3xy^{\frac{1}{2}}-3ac$	$5ab^3+4a^2c-bc^3$
$5x^3y-3bc+ab$	$4xy^{\frac{1}{2}}-5ac$	$3ab^3+5a^2c-2bc^3$
$7x^3y-\frac{1}{2}bc+4ab$	$3xy^{\frac{1}{2}}-2ac$	$8ab^3+8a^2c-8bc^3$
$8x^3y-2\frac{1}{2}bc+3ab$	$8xy^{\frac{1}{2}}-ac$	$4ab^3+3a^2c-7bc^3$
$3x^3y-7bc+6ab$	$5xy^{\frac{1}{2}}-7ac$	$ab^3+6a^2c-3bc^3$
$4x^3y-3bc+2ab$	$3xy^{\frac{1}{2}}-2ac$	$5ab^3+5a^2c-4bc^3$
$5x^3y-4bc+3ab$	$4xy^{\frac{1}{2}}-13ac$	$2ab^3+a^2c-5bc^3$

CASE 2.

26. *When the quantities are alike, but have unlike signs.*

Take the difference between the sum of the positive and the sum of the negative co-efficients, prefix the sign of the greater, and annex the literal part.

* The sign + or —, is frequently prefixed to quantities which are not preceded by others, to or from which they are required to be added or subtracted; a preceding quantity, however, may always be supposed. Algebraic quantities are said to be positive, or negative, according as they are preceded by the signs + or —; the former being always understood where none is expressed.

EXAMPLES.

$4ab^2-3cd^2$	x^2y^3	$5a^3b-7x^2y^3$
$2ab^2+6cd^2$	$-3x^2y^3$	$-3a^3b+4x^2y^3$
$-5ab^2+4cd^2$	$5x^2y^3$	$4a^3b+2x^2y^3$
ab^2-7cd^2	$7x^2y^3$	$-8a^3b+x^2y^3$
$-3ab^2+5cd^2$	$-4x^2y^3$	$7a^3b-3x^2y^3$
$6ab^2-cd^2$	$-9x^2y^3$	$-a^3b+5x^2y^3$
$5ab+4cd^2$	$-3x^2y^3$	

$3ab^3-5c^2d^2$	$8\sqrt{ay}$	$20ax-7b\sqrt{xy}$
$-5ab^3+2c^2d^2$	$-9\sqrt{ay}$	$-31ax+5b\sqrt{xy}$
$-ab^3+4c^2d^2$	$-7\sqrt{ay}$	$27ax-6b\sqrt{xy}$
$7ab^3-3c^2d^2$	$3\sqrt{ay}$	$18ax-5b\sqrt{xy}$
$8ab^3-5c^2d^2$	$5\sqrt{ay}$	$-4ax+13b\sqrt{xy}$
$-6ab^3+7c^2d^2$	$-4\sqrt{ay}$	$-30ax+11b\sqrt{xy}$

CASE 3.

27. *When there are unlike quantities, and different signs.*

Collect the like quantities as in the foregoing cases, and connect the results by their proper signs.

EXAMPLES.

$4ax-130+3x^{\frac{1}{2}}$	$7ax$	$3a^2-7bc+5ad$
$5x^2+3ax+9x^2$	$-5xy$	$2xy-7ad-5a^2$
$7xy-4x^{\frac{1}{2}}+90$	$-3ax$	$8ad-6xy-3bc$
$6x^2-5\sqrt{x}+3xy$	$2xy$	$5xy+2a^2-4ad$
$40-5ax-10xy$	$-3ab$	$4bc+3ad-xy$
$2ax+20x^2-6x^{\frac{1}{2}}$	$4ax-3xy-3ab$	

$5x^2y - 7ax - 3bc$	$14ax - 2x^2$
$2bc + 3x^2y + 4ax$	$5ax + 3xy$
$3ax - 5bc + 7ad^3$	$8y^2 - 4ax$
$2bc + 3ad^2 - 6ax$	$3x^2 + 26$
$x^2y - 5bc + 4ax$	$4xy - 5y^2$
$8ad^2 - 3ax - 4x^2y$	$5x^2 - 8ax$
$ax - 18ad^2 - 3bc$	$7y^2 - 26$

$x^3y^2 + 4ad^3 - 7b^2c^2$	$ax^3y^2 + 3ab^3y^3 - 5c^2x^2y^2 + b^2x^2y^2$
$5b^2c^2 - 7ad^3 + 2ac^4$	$3b^2x^2y^2 - 4ax^3y^2 + 4a^2xy^3 - 5a\,b^2y^3$
$3ac^4 + 6b^2c^2 + 5x^3y^2$	$2ab^2y^3 - 3b^2x^2y^2 - 2ax^3y^2 + 2c^2x^2y^2$
$8ad^3 - 5ac^4 - 6b^2c^2$	$4c^2x^2y^2 + 2ax^3y^2 - 3b^2x^2y^2 + 4ab^2y^3$
$5x^3y^2 + ad^3 - b^2c^2$	$2b^2x^2yz + 4ab^2y^3 + 2ax^3y^2 - 2c^2x^2y^2$
$3b^2c^2 - 5x^3y^2 - 2ac^4$	$3c^2x^2y^2 - 2ax^3y^2 + 5b^2x^2yz + 9ab^2y^3$
$6x^3y^2 - 2ac^4 - 6ad^3$	$4ab^2y^3 - 3c^2x^2y^2 - ax^3y^2 - 7b^2x^2yz$

1. Add $3a+b$, $7a-5b$, $7b-3a$, $4a-2b$, and $8a-5b$ together. Result, $19a-4b$.

2. Add $5ax+7bc-3a^2$, $8bc+4a^2-2ax$, $11a^2-4bc-6ax$ and $2ax-3bc+6a^2$ into one sum.

Result, $8bc+18a^2-ax$.

3. Add together the following quantities, $7abc+5a^2d-3xy^2$, $2a^2d+5xy^2-3abc$, $9bc^3+8xy^2+3a^2d$, $4bc^3-3abc-a^2d$, and $a^2d-10xy^2-abc-7bc^3$.

Result, $10a^2d+6bc^2$.

SUBTRACTION.

28. Change the signs of the quantities which are to be subtracted, or conceive them to be changed, then collect the terms as in addition.

EXAMPLES.

From $8xy-3+6x-y$	$3x^2-2ab$	$5ab^2-7b^2c+ad$
Take $3xy-7-6x+5y$	x^2-7ab	$3ab^2-2b^2c-4ad$
$5xy+4+12x-6y$	$2x^2+5ab$	$2ab^2-5b^2c+5ad$

From $17ab-3xy+5b^2$	$8ax-5bc$	$9aey+4bc$
Take $11ab+4xy+7b^2$	$7ax-7bc$	$2aey+7bc$
$6ab-7xy-2b^2$	$ax+2bc$	$7aey-3bc$.

From $7x^2-2a\sqrt{x}-18+5bc$	$4ad-7xy+3bc$
Take $3x^2-7a\sqrt{x}-9-3bc$	$5ad-y^2z+5bc$
$4x^2+5a\sqrt{x}-9+8bc$	$-ad-7xy-2bc+$

1. From $a+b$ take $a-b$. Result, $2b$.
2. From $8a-12x$ take $5a+3x$. Result, $3a-15x$.
3. From $12a+10b+13ax-3ab$, take $7a-5b+3ax$.
   Result, $5a+15b+10ax-3ab$.
4. From the sum of $3ab-7ax$ and $7ab+3ax$, take $4ab-3ax-4xy$. Result, $6ab-ax+4xy$.
5. From $5x^2-4xy+5$, subtract $4x^2-4xy+9$.
   Result, x^2-4.
6. From ax^3-bx^2+x subtract px^3-cx^2+ex.
   Result, $(a-p)\ x^3+(c-b)x^2+(1-e)x$.
7. From bx^3+cx^2-dx+e take px^3-qx^2+rx-s.
   Result, $(b-p)x^3+(c+q)x^2-(d+r)x+e+s$.

MULTIPLICATION.

Case I.

29. *When the factors are both simple quantities.*

Multiply the co-efficients together, and annex all the letters to the product.*

---

* From Def. 14, it is obvious, that the product of two or more powers of any letter, may be expressed by that letter with an index equal to the sum of the indices of the factors; thus, $a^3a^2=aaa.aa=a^5$.

Note.—When the signs of the factors are like, the product must be made positive; when unlike, negative.

EXAMPLES.

$10ab$	$2xy^3$	$37a^2b$	$5ad$
$5a^2c$	$-6x^2y$	$3ab^2c$	$3a^2b^3o$
$50a^3bc$	$-12x^3y^4$		

$-acx^2$	$3xyz$	$5a^3bc$	$-4ad^2$	$5x^3y^2$
$-bc^2x$	$-2x^2y$	$7ac^3$	$-5a^2b$	$-4x^2y^3$

CASE 2.

30. *When one of the factors is a compound quantity.*

Multiply the simple factor into each term of the compound ones, and connect the products by their proper signs.

EXAMPLES.

$a^2-2ab+b^2$	$xy+2y^2$	$2ab-2b^2$
a^2	$3ax$	$-4b^2$
$a^4-2a^3b+a^2b^2$		

xy^3+4bc^2-9	$8x^3y+5ay^3-7bc^2$	$3c^2+ax$
$5cx$	$-2a^2c$	$5ax$

CASE 3.

31. *When both factors are compound quantities.*

Multiply the whole multiplicand by every term of the multiplier, and collect the several products as in addition.

EXAMPLES.

$$\begin{array}{r} a+b \\ a+b \\ \hline a^2+ab \\ \quad ab+b^2 \\ \hline a^2+2ab+b^2 \end{array} \qquad \begin{array}{r} a^2+ab+b^2 \\ a^2-ab+b^2 \\ \hline a^4+a^3b+a^2b^2 \\ \quad -a^3b-a^2b^2-ab^3 \\ \qquad\qquad a^2b^2+ab^3+b^4 \\ \hline a^4 \qquad +a^2b^2 \qquad +b^4 \end{array}$$

$$\begin{array}{r} a+b \\ a-b \\ \hline \\ \hline \end{array} \qquad \begin{array}{r} x^2+xy+y^2 \\ x-y \\ \hline \\ \hline \end{array} \qquad \begin{array}{r} x^4+x^3y+x^2y^2+xy^3+y^4 \\ x-y \\ \hline \\ \hline \end{array}$$

$$\begin{array}{r} 2x^2+3xy-4y^2 \\ 3x-5y \\ \hline \\ \hline \end{array} \qquad \begin{array}{r} 2ab-3ac+4bc \\ ab+ac-bc \\ \hline \\ \hline \end{array} \qquad \begin{array}{r} x^2-xy+y^2 \\ x-y \\ \hline \\ \hline \end{array}$$

1. Multiply $5a^2+4b^2$ by $5a^2-4b^2$.
   Result, $25a^4-16b^4$.
2. Multiply $x^3+3x^2y+3xy^2+y^3$ by $x^2+2xy+y^2$.
   Result, $x^5+5x^4y+10x^3y^2+10x^2y^3+5xy^4+y^5$.
3. Multiply $3x^2+2xy+5y^2$ by x^2-xy+y^2.
   Result, $3x^4-x^3y+6x^2y^2-3xy^3+5y^4$.
4. Multiply $5x^2+10xy+5y^2$ by $5x^2-10xy+5y^2$.
   Result, $25x^4-50x^2y^2+25y^4$.
5. Multiply $a^3+3a^2b+3ab^2+b^3$ by $a^3-3a^2b+3ab^2-b^3$.
   Result, $a^6-3a^4b^2+3a^2b^4-b^6$.
6. Multiply a^2-ab+b^2-bc by a^2+ab-b^2+bc.
   Result, $a^4-a^2b^2+2ab^3-2ab^2c+2b^3c-b^2c^2-b^4$.
7. Multiply $x^4-x^3y+x^2y^2-xy^3+y^4$ by x^2+xy+y^2.
   Result, $x^6+x^4y^2-x^3y^3+x^2y^4+y^6$.

8. Multiply $x+y$ by $x-y$, and the product by x^2+y^2.
Result, x^4-y^4.

9. Find the continued product of $a^2-2ab+b^2$, $a^3+2ab+b^3$, $a^3+a^2b+ab^2+b^3$ and $a-b$.
Result, $a^8-2a^6b^2+2a^2b^6-b^8$.

10. Find the continued product of $3x+6$, $3x+2$, $3x-2$, and $3x-6$. Result, $81x^4-360x^2+144$.

11. What is the continued product of $3a-2b$, $4a-3b$, $4a+3b$, and $3a+2b$? Ans., $144a^4-145a^2b^2+36b^4$.

12. Multiply $a^5-a^4b+a^3b^2-a^2b^3+ab^4-b^5$, by $a+b$.
Result, a^6-b^6.

13. Multiply $x^3+x^2y+xy^2+y^3$, by $x^3-x^2y+xy^2-y^3$.
Result $x^6+x^4y^2-x^2y^4-y^6$.

When the factors consist of several terms with the powers regularly ascending, the multiplication may be effected by the co-efficients alone, and the literal part be afterwards supplied. Thus in the 5th example, we may take

$$\begin{array}{r} 1+3+3+1 \\ 1-3+3-1 \\ \hline 1+3+3+1 \\ -3-9-9-3 \\ 3+9+9+3 \\ -1-3-3-1 \\ \hline \end{array}$$

1 0—3 0+3 0—1. Then to supply the literal part, we observe, that a^3, $a^3=a^5$, the first term; and as the powers of a descend and those of b ascend in a regular order, the terms, without their co-efficients, must be, a^5, a^5b, a^4b^2, a^3b^3, a^2b^4, ab^5, b^6. But the 2d, 4th and 6th co-efficients are each $=0$, therefore the terms are $a^6-3a^4b^2+3a^2b^4-b^6$.

In like manner, the 7th example may be performed thus:

3

$$
\begin{array}{l}
1-1+1-1+1 \\
1+1+1 \\
\hline
1-1+1-1+^{1} \\
\quad 1-1+1-1+1 \\
\quad\quad 1-1+1-1+1 \\
\hline
1\quad 0+1-1+1\quad 0+1.
\end{array}
$$

In which the 2d and 6th co-efficients are $=0$, hence the 2d and 6th terms x^5y and xy^5, do not appear in the product.

DIVISION.

Case I.*

32. *When the divisor and dividend are both simple quantities.*

Divide the co-efficient of the dividend by that of the divisor; expunge from the dividend such letters as are common to it and the divisor, when they have the same exponents; when the exponents are not the same, subtract the exponent of the divisor from that of the dividend, and use the remainder as the index of the letter in the quotient; write in the quotient the letters which have not been expunged, with the co-efficient above determined.

Note.—In division, as in multiplication, like signs in the divisor and dividend require the positive sign in the quotient; unlike signs, the negative.

* If the student, who is just entering into this science, should express the powers by repetitions of the letters, the process will be more simple, and the grounds of the method indicated by the rule be rendered obvious.

EXAMPLES.

$3ab)9a^2bcd$ $\quad -5x^2y^3)15x^4y^3z^2$ $\quad 2ad)6a^3d^2c$

$3acd$ $\quad -3x^2z^2$

$17ab^2)51a^2b^5$ $\quad 3a^2c)12a^4bc^3$ $\quad 7a^2b)-21a^4bc^2$

$-3axz)-18a^2xy^2z^2$ $\quad 15abc^2)-45a^2b^3c^4$

33. *When the dividend is not a multiple of the divisor.*

Set them down as a vulgar fraction, the divisor being the denominator; and divide the terms by such numbers and quantities as are common to both; the result will be the fractional answer.

$$\frac{15a^2bc}{10ab^2}=\frac{3ac}{2b} \qquad \frac{18axy^2}{-8ax^3y}=\frac{-9y}{4x^2}$$

1. Divide $28xy^3$ by $7x^2y^2$. Result, $\frac{4y}{x}$
2. Divide $-10abc$ by $15abd$ Result, $\frac{-2c}{3d}$
3. Divide $11mn$ by $22m^3$ Result, $\frac{n}{2m^2}$
4. Divide $19xyz$ by $25x^2yz^2$ Result, $\frac{19}{25xz}$

CASE 2.

34. *When the dividend is a compound quantity, and the divisor a simple one.*

Divide every term of the dividend by the divisor, as in

the former case, and connect the results by their proper signs.

EXAMPLES.

$$\frac{18ax^2-15a^2y+27a^3}{3a}=6x^2-5ay+9a^2$$

2. Divide $15axy^2-30b^2xy-10x^2y^2$ by $5xy$.
3. Divide $30a^2c-15ac^2+18ac$ by $3ac$.
4. Divide $15a^2bc-15acx^2+5ad^2$ by $-5ac$.
5. Divide $21a^3x^3-7a^2x^2-14ax$ by $7ax$.
6. Divide $72a^2b^2y^2-12a^3by+36ab^3y$ by $6aby$.
7. Divide $28x^3y^2+35x^2y^4-7x^2y^4z$ by $7x^2y^2$.

CASE 3.

35. *When the divisor and dividend are compound quantities.*

Arrange the terms, both of the divisor and dividend, in such manner that the higher powers of some one letter shall always precede the lower; then find the first term of the quotient, by using the first of the divisor and dividend, as in the first case. Multiply the whole divisor by the quotient thus found; subtract the product from the dividend, as in common arithmetic; and proceed till the dividend is exhausted. The remainder, if any thing remain, with the divisor for a denominator, must be annexed, with its proper sign, to the quotient.

EXAMPLES.

$$\begin{array}{l}a+b)a^3+3a^2b+3ab^2+b^3(a^2+2ab+b^2\\ \quad\ \ a^3+\ a^2b\\ \hline \qquad\qquad 2a^2b+3ab^2\\ \qquad\qquad 2a^2b+2ab^2\\ \hline \qquad\qquad\qquad\quad ab^2+b^3\\ \qquad\qquad\qquad\quad ab^2+b^3\end{array}$$

$$a^2-2ab+b^2)a^5-5a^4b+10a^3b^2-10a^2b^3+5ab^4-3b^5(a^3-3a^2b+3ab^2-b^3$$
$$a^5-2a^4b+a^3b^2$$
$$-3a^4b+9a^3b^2-10a^2b^3$$
$$-3a^4b+6a^3b^2-3a^2b^3$$
$$3a^3b^2-7a^2b^3+5ab^4$$
$$3a^3b^2-6a^2b^3+3ab^4$$
$$-a^2b^3+2ab^4-3b^5$$
$$-a^2b^3+2ab^4-b^5$$
$$-2b^5$$

3. Divide $x^4-4x^3y+6x^2y^2-4xy^3+y^4$ by $x-y$.
Result, $x^3-3x^2y+3xy^2-y^3$.

4. Divide a^2-b^2 by $a-b$. Result, $a+b$.

5. Divide a^3+b^3 by $a+b$. Result, a^2-ab+b^2.

6. Divide $y^3-3y^2-4y+12$ by y^2-4. Result, $y-3$.

7. Divide $b^6-3b^4x^2+3b^2x^4-x^6$ by $b^3-3b^2x+3bx^2-x^3$.
Result, $b^3+3b^2x+3bx^2+x^3$.

8. Divide $24a^4-b^4$ by $3a-6b$.
Result, $8a^3+16a^2b+32ab^3+64b^3+\frac{383b^4}{3a-6b}$

9. Divide $a^4+4a^2b^2-32b^4$ by $a+2b$.
Result, $a^3-2a^2b+8ab^2-16b^3$.

10. Divide x^6+y^6 by $x+y$.
Result, $x^5-x^4y+x^3y^2-x^2y^3+xy^4-y^5+\frac{2y^6}{x+y}$.

11. Divide x^9-y^9 by $x-y$.
Result, $x^8+x^7y+x^6y^2+x^5y^3+x^4y^4+x^3y^5+x^2y^6+xy^7+y^8$.

12. Divide $a^4+4a^3b+6a^2b^2+4ab^3+b^4+4a^3c+12a^2bc+12ab^2c+4b^3c+6a^2c^2+12abc^2+6b^2c^2+4ac^3+4bc^3+c^4$ by $a^2+2ab+2ac+b^2+2bc+c^2$.
Result, $a^2+2ab+2ac+b^2+2bc+c^2$.

PROMISCUOUS EXAMPLES.

1. Required the sum of $5ac+7be$, $3ac-4be$, $7ac-3be$, $8ac+4be$, and $2ac-3be$. Ans $25ac+be$.

2. Collect $3ax+5xy-7z^2$, $3xy-2ax+4z^2$, $7ax+3z^2-xy$, and $5z^2-4ax+2xy$, into one sum; and subtract therefrom the sum of the following quantities, $3ax-5xy+4z^2$, $7xy-5ax-6z^2$, and $5ax-2xy+6z^2$.

Result, $ax+9xy+z^2$.

3. Required the product of $x^3+x^2y+xy^2+y^3$ by $x-y$.

Ans. x^4-y^4.

4. What is the sum of the products of $x^3+3x^2y+3xy^2+y^3$ by $x+y$, and $x^2-2xy+y^2$ by $x^2-2xy+y^2$?

Ans. $2x^4+12x^2y^2+2y^4$.

5. Divide x^5-y^5 by $x-y$, and from the quotient subtract the product of x^2+y^2 by x^2-xy.

Result, $2x^3y+2xy^3+y^4$.

6. Multiply $3a^3-9a^2b+9ab^2-3b^3$ by $a^2+2ab+b^2$, and $a^3+3a^2b+3ab^2+b^3$ by $3a^2-6ab+3b^2$, and find the sum of the products. Result, $6a^5-12a^3b^2+6ab^4$.

7. If a^4-b^4, be divided by $a-b$, and the product of a^2+ab+b^2 by $a-b$, subtracted from the quotient, what will the remainder be? Ans. $a^2b+ab^2+2b^3$.

8. Divide $x^8+8x^7y+28x^6y^2+56x^5y^3+70x^4y^4+56x^3y^5+28x^2y^6+8xy^7+y^8$ by $x^2+2xy+y^2$: and multiply $x^4-4x^3y+6x^2y^2-4xy^3+y^4$ by $x^2-2xy+y^2$, and find the sum and difference of the results.

Result, { Sum, $2x^6+30x^4y^2+30x^2y^4+2y^6$.
Difference, $12x^5y+40x^3y^3+12xy^5$.

9. Divide a^9-b^9 by $a^6+a^3b^3+b^6$; and multiply a^2-ab+b^2 by $a+b$; and find the sum and difference of the results. Sum, $2a^3$. Diff. $2b^3$.

When a co-efficient, either numerical or literal, is common to every term of the divisor or dividend, the process

may be shortened, by first dividing by that co-efficient, then using the quotient instead of the given quantity, and multiplying or dividing the result by such co-efficient, according as it was found in the dividend or divisor.

13. Divide $3ax^3-9ax^2y+9axy^2-3ay^3$ by $x^2-2xy+y^2$. Here the dividend $=3a\,(x^3-3x^2y+3xy^2-y^3)$. Hence if we divide $x^3-3x^2y+3xy^2-y^3$ by $x^2-2xy+y^2$, and multiply the resulting quotient $x-y$ by $3a$, we shall have $3ax-3ay$, the true quotient required.

14. Divide $5a^4-5b^4$ by a^2+b^2. Result, $5a^2-5b^2$.

15. Divide $7a^4-28a^3b+42a^2b^2-28ab^3+7b^4$ by $3a^2-6ab+3b^2$. Result, $\frac{7}{3}(a^2-2ab+b^2.)$

When the dividend and divisor contain no more than two letters, with regularly ascending and descending powers, the operation may frequently be abridged by omitting the letters, and proceeding with the co-efficients. Thus in the third example: page 29.

```
1—1) 1—4+6—4+1(1—3+3—1
     1—1
     ———
      —3+6
      —3+3
      ————
          3—4
          3—3
          ———
           —1+1
           —1+1
           ———
```

And we perceive from the first term of the dividend and divisor that the first term of the quotient is x^3; hence the result is readily discovered.

In the 10th example, the dividend, when the powers are regularly arranged, is $x^6+0x^5y+0x^4y^2+0x^3y^3+0x^2y^4+0xy^5-y^6$. Hence we may proceed in this manner.

$$
\begin{array}{l}
1+1)1+0+0+0+0+0--1(1--1+1--1+1--1 \\
\quad 1+1 \\
\quad \overline{} \\
\quad -1+0 \\
\quad -1--1 \\
\quad \overline{} \\
\qquad 1+0 \\
\qquad 1+1 \\
\qquad \overline{} \\
\qquad -1+0 \\
\qquad -1--1 \\
\qquad \overline{} \\
\qquad\quad 1+0 \\
\qquad\quad 1+1 \\
\qquad\quad \overline{} \\
\qquad\quad -1--1 \\
\qquad\quad -1--1 \\
\qquad\quad \overline{} \\
\qquad\qquad\quad \overline{}
\end{array}
$$

And as $\frac{x^6}{x}=x^5$, we readily discover the powers of x and y which must be annexed to these co-efficients.

SECTION II.

INVOLUTION, OR THE RAISING OF POWERS.

36. *To involve a given quantity to any power.**

Multiply the quantity by itself as many times as there are units in the index of the given power diminished by one.

EXAMPLES.

1. Required the 5th power of a^3b^2c.

$$(a^3b^2c)^5=a^{15}b^{10}c^5.$$

* When the quantity to be involved is a simple one, multiply its index, or indices, by that of the power proposed, observing that the even powers of negative quantities are positive, and the odd ones negative.

2. Required the 4th power of $2b-c$.

$$
\begin{array}{l}
2b\ -c \\
2b\ -c \\
\hline
4b^2-2bc \\
\quad -2bc\ +c^2 \\
\hline
4b^2-4bc\ +c^2 \\
2b\ -c \\
\hline
8b^3-\ 8b^2c+\ 2bc^2 \\
\quad -\ 4b^2c+\ 4bc^2-c^3 \\
\hline
8b^3-12b^2c+\ 6bc^2\ -c^3 \\
2b\ -c \\
\hline
16b^4-24b^3c+12b^2c^2-2bc^3 \\
\quad -8b^3c+12b^2c^2-6bc^3+c^4 \\
\hline
16b^4-32b^3c+24b^2c^2-8bc^3+c^4 \\
\hline
\end{array}
$$

3. Find the 3d power of $5a^2$.
4. Required the 5th power of $-2a$.
5. Required the 4th power of $a+x$.
6. Required the square of $3a-2b$.
7. Required the 6th power of $5a+b$.
8. Required the 3d power of $a+2b+c$.

37. A binomial is raised to any power, with great facility, by the following method.*

Set down, for the first term of the power, the first term of the root involved to the given power.

The succeeding terms, without the co-efficients, consist of the successive powers of the first term of the root, regularly descending, joined to the powers of the second, regu-

* The reasons for this process are given in a subsequent part of this work.—See Art. 98.

larly ascending from the first; the common difference of the indices being one.

The co-efficient of the second term is the index of the given power; and if a co-efficient, already found, be multiplied by the exponent of the leading quantity, or first term of the root, contained in the same term of the power, and divided by the number of terms to that place; the quotient will be the co-efficient belonging to the next term.*

Note.—To determine the signs of the different terms, it must be remembered that the even powers of negative quantities are positive, and the odd powers negative.

EXAMPLES.

Required the 5th power of $a+x$.

$a^5 \quad a^4x \quad a^3x^2 \quad a^2x^3 \quad ax^4 \quad x^5$

The co-efficients, 1, 5, $\frac{5\times4}{2}=10$, $\frac{10\times3}{3}=10$, etc.

Hence, $(a+x)^5=a^5+5a^4x+10a^3x^2+10a^2x^3+5ax^4+x^5$.

Required the 6th power of $2x-3y$.

$(2x)^6 \quad (2x)^5.3y \quad (2x)^4.(3y)^2 \quad (2x)^3.(3y)^3 \quad (2x)^2.(3y)^4$
$(2x).(3y)^5 \quad (3y)^6$

Or, $64x^6 \; 96x^5y \; 144x^4y^2 \; 216x^3y^3 \; 324x^2y^4 \; 486xy^5 \; 729y^6$.

The co-efficients 1, 6, $\frac{6\times5}{2}=15$, $\frac{15\times4}{3}=20$, 15, 6, 1.

Hence, $(2x-3y)^6=64x^6-576x^5y+2160x^4y^2-4320x^3y^3+4860x^2y^4-2916xy^5+729y^6$.

2. Required the 3d power of $a+y$.

Ans. $a^3+3a^2y+3ay^2+y^3$.

---

* The co-efficients of any two terms, equally distant, the one from the beginning, and the other from the end, are alike; hence, the computation of the first half of them will supply the whole.

3. Required the 4th power of $a-b$.

Ans. $a^4-4a^3b+6a^2b^2-4ab^3+b^4$.

4. Required the 7th power of $x+y$.

Result, $x^7+7x^6y+21x^5y^2+35x^4y^3+$, etc.

5. Required the 8th power of $a-c$.

Result, $a^8-8a^7c+28a^6c^2-56a^5c^3+70a^4c^4$, etc.

6. Required the 4th power of $2+x$.

Result, $16+32x+24x^2+8x^3+x^4$.

7. Required the 3d power of $a-3b$.

Result, $a^3-9a^2b+27ab^2-27b^3$.

This method of involution is easily extended to trinomials, quadrinomials, etc., by considering two or more of the terms as a single compound one.

8. Required the 4th power of $a+b-c$.

Here considering $b-c$ as the second term of the binomial, we have $(a+b-c)^4=a^4+4a^3(b-c)+6a^2(b-c)^2+4a(b-c)^3+(b-c)^4$; and involving $b-c$, by the same method, to the powers indicated, we have $(b-c)^2=b^2-2bc+c^2$; $(b-c)^3=b^3-3b^2c+3bc^2-c^3$; $(b-c)^4=b^4-4b^3c+6b^2c^2-4bc^3+c^4$; consequently, $(a+b-c)^4=a^4+4a^3b-4a^3c+6a^2b^2-12a^2bc+6a^2c^2+4ab^3-12ab^2c+12abc^2-4ac^3+b^4-4b^3c^2+6b^2c^2-4bc^3+c^4$.

9. Required the 3d power of $x+y+z$.

Result $x^3+3x^2y+3xy^2+y^3+3x^2z+6xyz+3y^2z+3xz^2+3yz^2+z^3$.

10. Required the second power of $a+b-c-d$.

Result, $a^2+2ab+b^2-2ac-2bc+c^2-2ad-2bd+2cd+d^2$.

11. Required the 4th power of $x+y-3z$.

Result, $x^4+4x^3y+6x^2y^2+4xy^3+y^4-12x^3z-36x^2yz-36xy^2z-12y^3z+54x^2z^2+108xyz^2+54y^2z^2-108xz^3-108yz^3+81z^4$.

12. Required the 5th power of $a+b+c$.

Result, $a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$
$+5a^4c+20a^3bc+30a^2b^2c+20ab^3c+5b^4c+$
$10a^3c^2+30a^2bc^2+30ab^2c^2+10b^3c^3+10a^2c^3$
$+20abc^3+10b^2c^3+5ac^4+5bc^4+c^5$.

13. Required the 3d power of $a+b-c-d$.

Result, $a^3+3a^2b+3ab^2+b^3-3a^2c-3a^2d-6abc$
$-6abd-3b^2c-3b^2d+3ac^2+3bc^2+6acd+$
$6bcd+3ad^2+3bd^2-c^3-3c^2d-3cd^2-d^3$.

EVOLUTION, OR THE EXTRACTION OF ROOTS.

Case I.

38. *To extract the root of a simple quantity.*

Extract the root of the co-efficient, for the numeral part; and divide the index or indices of the literal part, for the exponents of the root.*

When the root cannot be extracted, it must be indicated as in definition 16.

EXAMPLES.

1. Required the 4th root of $256a^4x^8$.

$\sqrt{256}=\pm 4$, the co-efficient; $a^{\frac{4}{4}}x^{\frac{8}{4}}=ax^2$, the literal part.

Hence, the root required is $4ax^2$, or $-4ax^2$.

2. What is the 5th root of $-5a^{15}b^{10}$?

The 5th root of 5 is a surd, and must be indicated thus: $\sqrt[5]{\ }5$ or $5^{\frac{1}{5}}$.

Hence, the root required is $-5^{\frac{1}{5}}a^3b^2$ or $-a^3b^2\sqrt[5]{5}$.

---

* The odd roots have the same signs as their powers; but the even roots of positive quantities may be either positive or negative. The even roots of negative quantities are impossible. These impossible or imaginary roots frequently become the subject of important investigations, as will appear in the sequel of this work.

3. Required the square root of $16a^2b^4$.

Result, $\pm 4ab^2$.

4. What is the square root of $625x^4y^8$?

Ans. $\pm 25x^2y^4$.

5. Required the cube root of $-125a^6b^3$.

Result, $-5a^2b$.

6. What is the 5th root of $-32a^{10}b^5c^{15}$?

Ans. $-2a^2bc^3$.

7. What is the 3d root of $7x^3y^3$? Ans. $x\sqrt[3]{7y^3}$.

8. What is the 4th root of $81a^4b^6$?

Ans. $\pm 3a\sqrt{b^3}$, or $\pm 3ab\sqrt{b}$.

9. Required the 5th root of $-243a^5b^6$.

Result, $-3ab\sqrt{b}$.

Case 2.

39. *To extract the square root of a compound quantity.*

Arrange the terms according to the dimensions of some letter, beginning with the highest.

Take the square root of the first term for the first term of the root, and subtract its square from the given quantity. Double the root thus found, for a defective divisor, divide the first term of the above remainder by this defective divisor, and annex the result both to the root and to the divisor.

Multiply the divisor thus completed by the term of the root last obtained, and subtract the product from the former remainder.

Divide the first term of the remainder as before, for the next term of the root; add this last and the preceding term of the root to the last complete divisor for a new divisor. Multiply, subtract, and proceed as before.

Note.—In these additions, regard must always be paid to the signs of the quantities.

EXAMPLES.

$$4a^2+12ab+9b^2-20ac-30bc+25c^2(2a+3b-5c$$
$$4a^2$$

$4a+3b$		$12ab+9b^2$
	$3b-5c$	$12ab+9b^2$
$4a+6b-5c$		$-20ac-30bc+25c^2$
		$-20ac-30bc+25c^2$

2. Required the square root of $x^2+2xy+y^2$.
Result, $x+y$.

3. Required the square root of $a^4+4a^3+6a^2+4a+1$.
Result, a^2+2a+1.

4. Required the square root of $x^4-4ax^3+6a^2x^2-4a^3x+a^4$.
Result, $x^2-2ax+a^2$.

5. What is the square root of $16x^4+24x^3+89x^2+60x+100$.
Ans. $4x^2+3x+10$.

6. What is the square root of $4a^4-12a^3x+5a^2x^2+6ax^3+x^4$?
Ans. $2a^2-3ax-x^2$.

7. What is the 4th root of $a^4+12a^3b+54a^2b^2+108ab^3+81b^4$?*
Ans. $a+3b$.

8. What is the 4th root of $a^8-4a^6b^2+6a^4b^4-4a^2b^6+b^8$?
Ans. a^2-b^2.

CASE 3.

40. *To extract any root of a compound quantity.*

Arrange the terms as before directed; take the root of the first term for the first term of the root, and subtract its power from the given quantity.

Take for a divisor twice this root, three times its square, four times its third power, five times its fourth power, etc., according as the 2d, 3d, 4th, 5th, etc. roots are required to be extracted; divide the first term of the remainder, and annex the result to the root before found.

* The 4th, 8th, 16th, etc. roots, may be obtained by 2, 3, 4, etc. extractions of the square root.

Involve the whole root, thus obtained, to the given power, and subtract from the given quantity.

Divide the first term of the last remainder by the former divisor, annex the result to the root, and proceed as before.

EXAMPLES.

1. Required the 3d root of $8a^3+36a^2b+54ab^2+27b^3-48a^2c-144abc-108b^2c+96ac^2+144bc^2-64c^3$.

$8a^3+36a^2b+54ab^2+27b^3-48a^2c-144abc-108b^2c+96ac^2+144bc^2-64c^3(2a+3b-4c$

$8a^3$

$12a^2)\ 36a^2b$

$8a^3+36a^2b+54ab^2+27b^3=(2a+3b)^3$

$12a^2)\ -48a^2c$

$8a^3+36a^2b+54ab^2+27b^3-48a^2c-144abc-108b^2c+96ac^2+144bc^2-64c^3=2a+3b-4c)^3$

The reason of this process, as well as of that laid down under the former case, is readily seen if we consider that in the involution of powers, the second term of the power consists of the continued product of the index of the power, the first term of the root raised to the next inferior power, and the second term of the root. Thus, if we raise $a+b$ to the 2d, 3d, 4th, etc. power, we find the second term $2ab$, $3a^2b$, $4a^3b$, etc., and the first term of the power consists of the first term of the root involved to the power in question.

2. Required the 3d root of $27x^3-54x^2y+36xy^2-8y^3$.
Ans. $3x-2y$.

3. What is the 3d root of $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$? Ans. x^2-2x+1.

4. Required the 5th root of $32x^5-80x^4+80x^3-40x^2+10x-1$. Result, $2x-1$.

5. Required the 4th root of $16a^4-96a^3x+216a^2x^2-216ax^3+81x^4$. Ans. $2a-3x$.

6. Find the 5th root of $x^{10}+10x^9+35x^8+40x^7-30x^6-68x^5+30x^4+40x^3-35x^2+10x-1$.
Result, x^2+2x-1.

41. *Note.*—The roots of compound quantities may sometimes be obtained, with great facility, by a method of trial. To effect which, it may be observed, that the power of a binomial consists of as many terms as there are units in the exponent, increased by one. Hence, if the number of terms of a quantity, whose root is to be extracted, exceeds this sum, we may conclude that the root consists of more terms than two.

When the root is judged to be a binomial, take the roots of the extreme terms, and connect the results by the sign + or —; but when the root appears to contain more terms than two, take also the roots of one or more of the other given terms, and connect the roots thus obtained by the signs + or —, as may be judged proper.

Involve this supposed root to the given power, and if

the result agrees with the quantity given, the root is manifestly correct.

This expedient, however, is not likely to abridge the labor of a student in the early periods of his course.

Section III.

ALGEBRAIC FRACTIONS.

42. Algebraic fractions depend upon the same principles as those in common arithmetic, and are managed by similar rules, proper regard being paid to the signs and algebraic modes of operation.

EXAMPLES.

1. Required the greatest common divisor of x^3-b^2x, and $x^2+2bx+b^2$.*

$$
\begin{array}{l}
x^3-b^2x)x^2+2bx+b^2(1 \\
\quad x^2-b^2 \quad x^2-b^2 \\
\hline
\qquad 2bx+2b^2 \\
\qquad x+b)x^2-b^2(x-b \\
\qquad\qquad x^2+bx \\
\qquad\qquad \hline
\qquad\qquad -bx-b^2 \\
\qquad\qquad -bx-b^2
\end{array}
$$

Hence, $x+b$ is the divisor required.

---

* In examples of this kind, we are frequently obliged to adopt expedients which are not usual in common arithmetic. When every term of the divisor has a factor, which does not contain a letter or number common to every term of the dividend, we divide by it, and use the quotient as a divisor. When the divisor has been thus reduced, it often happens, that the division is still impracticable; in

2. Required to reduce $\frac{3a^2-2a-1}{4a^3-2a^2-3a+1}$ to its lowest terms.*

First to find a common divisor.

$3a^2-2a-1)\ 4a^3-2a^2-3a+1\ (4a$

3

$12a^3-6a^2-9a+3$

$12a^3-8a^2-4a$

$2a^2-5a+3$

3

$6a^2-15a+9\ (2$

$6a^2-4a-2$

$-11a+11$

---

which case, we multiply the dividend by such a number as will make its first term a multiple of the first of the divisor.

By these processes we make no change in the common divisor, unless we divide by a number or quantity which is common to the given terms; or introduce into one of them a divisor of the other.

* Solutions may sometimes be facilitated, by resolving the quantities concerned into their component factors. In this example the numeratoris evidently $=3a^2-3a+a-1=3a(a-1)+a-1=(3a+1)(a-1)$; and the deuominator $=4a^3-4a^2+2a^2-2a-a+1=4a^2(a-1)+2a(a-1)-1(a-1)=(4a^2+2a-1)(a-1)$. Hence $a-1$ is manifestly the common divisor, and $\frac{3a+1}{4a^2+2a-1}$ the required result.

In example 3, we have the numerator $=a^3(a^2-b^2)$, and from example 8, page 25, it is seen that $a^4-b^4=(a^2-b^2)(a^2+b^2)$. Hence, a^2-b^2 is the common divisor; and $\frac{a^5-a^3b^2}{a^4-b^4}=\frac{a^3}{a^2+b^2}$.

$$\begin{array}{r|l} -11a+11 & 3a^2-2a-1(3a+1 \\ \text{Com. div. } a-1 & 3a^2-3a \\ \hline & \quad a-1 \\ & \quad a-1 \\ \hline \end{array}$$

$$a-1 \;\Big|\; \frac{3a^2-2a-1}{4a^3-2a^2-3a+1}=\frac{3a+1}{4a^2+2a-1} \text{ Result.}$$

3. Reduce $\frac{a^5-a^3b^2}{a^4-b^4}$ to its lowest terms. Result, $\frac{a^3}{a^2+b^2}$

4. Reduce $\frac{x^2-1}{xy+y}$ to its lowest terms. Result, $\frac{x-1}{y}$

5. Reduce $\frac{5a^5+10a^4b+5a^3b^2}{a^3b+2a^2b^2+2ab^3+b^4}$ to its lowest terms.

Result, $\frac{5a^4+5a^3b}{a^2b+ab^2+b^3}$

6. Reduce $\frac{a^4-x^4}{a^3-a^2x-ax^2+x^3}$ to its lowest terms.

Result, $\frac{a^2+x^2}{a-x}$

7. Reduce $\frac{6a^2+7ax-3x^2}{6a^2+11ax+3x^2}$ to its lowest terms.

Result, $\frac{3a-x}{3a+x}$

8. Reduce $\frac{15x^3-x^2-x+3}{9x^4+2x^2+1}$ to its lowest terms.

Result, $\frac{5x+3}{3x^2+2x+1}$

9. Reduce $\frac{a}{2b}, \frac{2c}{3d}, \frac{3x}{4a}$, to other equivalent fractions having a common denominator.*

Result, $\frac{6a^2d}{12abd}, \frac{8abc}{12abd}, \frac{9bdx}{12abd}$

10. Reduce $\frac{3b}{4a^2}, \frac{2c}{3a}$, and $\frac{d}{2a}$ to a common denominator.

Result, $\frac{9b}{12a^2}, \frac{8ac}{12a^2}, \frac{6ad}{12a^2}$

11. Reduce $\frac{1}{3}, \frac{3a^3}{4}$, and $\frac{2a^2+b^2}{a+b}$, to a common denominator.

Result, $\frac{4a+4b}{12a+12b}, \frac{9a^3+9a^2b}{12a+12b}, \frac{24a^2+12b^2}{12a+12b}$

---

* The *least* common denominator should always be found in examples of this kind.

To find the least common multiple of two or more numbers or quantities, divide such of them as have a common divisor, by the greatest prime number they contain. Repeat the process on the quotients and undivided quantities as long as a divisor common to two of them can be found; then multiply these divisors, the several final quotients, and undivided quantities continually together, the product is the common multiple required. For such product must contain all the factors included in the given quantities, and none of them are repeated oftener than in the given terms. Thus to find the least common multiple of $4a^3$, $3ab$, $6b^2$, $10bc$. The only common measures of the co-efficients are 2 and 3, and the literal divisors, common to two or more, are a and b. Arranging them thus:

$2)4a^3,\ 3ab,\ 6b^2,\ 10bc$

3	$2a^3$	$3ab$	$3b^2$	$5bc$
a	$2a^3$	ab	b^2	$5bc$
b	$2a^2$	b	b^2	$5bc$

$2a^2 \times 1 \times b \times 5c \times 2 \times 3ab = 60a^3b^2c$ the quantity sought.

12. Reduce $\frac{31}{40}, \frac{2x}{3}, \frac{a^2+2x}{a}$, to a common denominator.

Result, $\frac{93a}{120a}, \frac{80ax}{120a}, \frac{120a^2+240x}{120a}$

13. Reduce $a-\frac{2x}{3a}$ to an improper fraction.

Result, $\frac{3a^2-2x}{3a}$

14. Reduce $10+\frac{2x-8}{3x}$ to an improper fraction.

Result, $\frac{32x-8}{3x}$

15. Required an improper fraction equivalent to $1+2x-\frac{x-5}{7x}$.

Result, $\frac{14x^2+6x+5}{7x}$

16. What is the equivalent improper fraction to $a-\frac{z^2-a^2}{a}$?

Ans. $\frac{2a^2-z^2}{a}$

17. Reduce $\frac{3ax+4x^2}{a+x}$ to a whole or a mixed quantity.

Result, $3x+\frac{x^2}{a+x}$

18. Reduce $\frac{2a^2+2b^2}{a-b}$ to a whole or mixed quantity.

Result, $2a+2b+\frac{4b^2}{a-b}$

19. Reduce $\frac{27a^3-3b^2-4x+9a^2}{9a^3}$ to a mixed quantity.

Result, $3a+1-\frac{3b^2+4x}{9a^3}$

20. Reduce $\frac{x^3+3ax^2-a^3-b}{x^3+a^3}$ to a whole or mixed quantity.

Result, $x^3-a^3+\frac{3ax^2-b}{x^3+a^3}$

21. Add $\frac{a}{2b}$, $\frac{2a}{3b}$ and $\frac{5b}{6a}$ together. Sum, $\frac{7a^2+5b^2}{6ab}$

22. Required the sum of $a-\frac{3x^2}{b}$ and $b+\frac{2ax}{c}$.

Result, $a+b+\frac{2abx-3cx^2}{bc}$

23. What is the sum of $2a+\frac{a+3}{5}$, and $4a+\frac{2a-5}{4}$?

Ans. $6a+\frac{14a-13}{20}$

24. Add $\frac{a+b}{a-b}$ to $\frac{a-b}{a+b}$. Sum $\frac{2a^2+2b^2}{a^2-b^2}$ or $2+\frac{4b^2}{a^2-b^2}$

25. What is the sum of $2a$, $\frac{a}{a-x}$ and $\frac{a-x}{a+x}$?

Ans. $2a+2+\frac{3x^2-ax}{a^2-x^2}$

26. From $\frac{3a}{4}$ subtract $\frac{4a}{7}$ Remainder, $\frac{5a}{28}$

27. From $\frac{3ax}{4}$ take $\frac{2ax}{3}$. Rem. $\frac{ax}{12}$

28. From $\frac{2a-b}{4c}$ subtract $\frac{3a-4b}{3b}$.

Rem. $\frac{6ab-3b^2-12ac+16bc}{12bc}$

29. From $4a+\frac{2a-b}{3c}$ subtract $2a-\frac{4b-3a}{5a+c}$.

Rem. $2a+\frac{10a^2-5ab-7ac+11bc}{15ac+3c^2}$

30. Subtract $\frac{a+x}{a(a-x)}$ from $a+\frac{a-x}{a(a+x)}$. Rem. $a-\frac{4x}{a^2-x^2}$

31. Required the continued product of $\frac{a}{x}$, $\frac{2x}{9a}$ and $\frac{a-x}{a+x}$. Product, $\frac{2a-2x}{9a+9x}$

32. Required the product of $\frac{a+1}{ab+c^2}$ and $\frac{ac-c}{a^2-1}$.

Prod. $\frac{c}{ab+c^2}$

33. Required the continued product of $a-\frac{b^2}{a}$, $\frac{a}{a+b}$, $\frac{3x}{a-b}$, and $\frac{7b}{bx-x^2}$. Prod. $\frac{21b}{b-x}$

34. Required the continued product of $\frac{a^2-x^2}{a+b}$, $\frac{a^2-b^2}{ax+x^2}$, and $x+\frac{ax}{a-x}$. Prod. $2a^2-ax-2ab+bx$

35. Divide $\frac{34a}{65b}$ by $\frac{17ax}{26bc}$. Quotient, $\frac{4c}{5x}$

36. Divide $\frac{26x^2}{35ay}$ by $\frac{13x}{25yz}$. Quot. $\frac{10xz}{7a}$

37. Divide $a-\frac{a^2-ax}{a+x}$ by $a+\frac{a^2+ax}{a-x}$. Quot. $\frac{ax-x^2}{a^2+ax}$

38. Divide $\frac{a-x}{a+x}$ by $\frac{a^2-x^2}{a^2+2ax+x^2}$. Quot. 1

39. Divide $\frac{3x-1}{9}$ by $\frac{12x+4}{27x}$. Quot. $\frac{3x}{4}$

40. Divide $\frac{2a-b}{12cd}$ by $\frac{5ac}{6d}$. Quot. $\frac{2a-b}{10ac^2}$

41. Divide $\frac{5a^4-5b^4}{2a^2-4ab+2b^2}$ by $\frac{5a^2+5ab}{4a-4b}$.*

Quot. $2a+\frac{2b^2}{a}$

42. Divide $\frac{x^4-b^4}{x^2-2bx+b^2}$ by $\frac{x^2+bx}{x-b}$. Quot. $x+\frac{b^2}{x}$

43. Divide $\frac{a^3-x^3}{a+x}$ by $\frac{a-x}{a^2+2ax+x^2}$.

Quot. $a^3+2a^2x+2ax^2+x^3$.

44. Divide $x^4-\frac{13}{6}x^3+x^2+\frac{4}{3}x-2$ by $\frac{4}{3}x-2$.

Quot. $\frac{3}{4}x^3-\frac{1}{2}x^2+1$.

---

* By inverting the divisor, and resolving the terms into their component factors we have—

$$\frac{5.(a^2+b^2)(a+b)(a-b)}{2(a-b)\ (a-b)} \times \frac{4(a-b)}{5a(a+b)}$$

Whence by expunging the corresponding factors, from the numerator and denominator we find the fraction reduced to $\frac{2(a^2+b^2)}{a} = 2a+\frac{2b^2.}{a}$

In like manner—

$$\frac{x^4-b^4}{x^2-2bx+b^2} \times \frac{x-b}{x^2+bx} = \frac{(x^2+b^2)(x+b)(x-b).(x-b)}{(x-b)(x-b).x(x+b)} = \frac{x^2+b^2}{x}$$
$$= x+\frac{b^2}{x}$$

45. Required the 4th power of $\frac{-2x}{3a}$. Result, $\frac{16x^4}{81a^4}$

46. Required the 3d power of $\frac{a-b}{a+b}$.

Result $1-\frac{6a^2b+2b^3}{a^3+3a^2b+3ab^2+b^3}$

47. What is the 4th power of $1-\frac{2x}{a}$?

Ans. $1-\frac{8x}{a}+\frac{24x^2}{a^2}-\frac{32x^3}{a^3}+\frac{16x^4}{a^4}$

48. What is the cube root of $\frac{64x^3}{27a^6b^3}$? Ans. $\frac{4x}{3a^2b}$

49. Required the square root of $\frac{9a^6}{4x^2y^4}$.

Result, $\pm\frac{3a^3}{2xy^2}$

50. Required the square root of $x^4-3x^3+\frac{11}{4}x^2-\frac{3}{4}x+\frac{1}{16}$. Result, $x^2-\frac{3}{2}x+\frac{1}{4}$

51. What is the 3d root of $\frac{1}{8}x^6-\frac{1}{2}x^5+\frac{41}{48}x^4-\frac{43}{54}x^3+\frac{41}{96}x^2-\frac{1}{8}x+\frac{1}{64}$? Ans. $\frac{1}{2}x^2-\frac{2}{3}x+\frac{1}{4}$

52. Required the square root of a^2+2b.

Result, $a+\frac{b}{a}-\frac{b^2}{2a^3}+\frac{b^3}{2a^5}-\frac{5b^4}{8a^7}$ etc.

Section IV.

EQUATIONS.

43. An equation is an algebraic expression, indicating that one quantity, or combination of quantities, is equal to another. The quantity, or combination, which stands on either side of the sign of equality, is called a member of the equation.

Thus, $ax+bx=ab+cd$, is an equation of which $ax+bx$ is the first member, and $ab+cd$ the second.

44. In the application of algebra to the solution of problems, the quantities concerned, both those which are given, and those which are required, are usually represented by symbols, and the conditions of the problem translated into the language of algebra. One or more equations are thus formed, including unknown as well as known quantities.

45. The number of independent equations required for the solution of a problem, is equal to the number of unknown quantities employed.

46. A *simple equation* is one which includes only the first or single power of the unknown quantity or quantities, as $ax+b=cx-d$.

47. A *quadratic equation* is one which includes the square of the unknown quantity. An equation which contains no power of the unknown quantity, but the square, is called *a simple quadratic*, as $ax^2=b$; but when the square and simple power of the unknown quantity are both included, it is called an *adfected quadratic equation;* as $x^2-ax=bc$.

48. A *cubic equation* is one which contains the cube of the unknown quantity; as $x^3+ax^2+bx=c$.

49. A *biquadratic equation* is one which contains the fourth power of the unknown quantity; as $x^4+a^2x^2+c^3x=b^2d^2$.

50. Equations, in general, are said to be of as many

dimensions as there are units in the index of the highest power of the unknown quantity.

51. In general, an equation which contains but one power of the unknown quantity is called a *pure equation;* thus, $x^3=b+c$, is called a pure quadratic equation; $x^3=b$, a pure cubic equation, etc.

Great part of the business of algebra consists in deducing, from given equations, the values of the unknown quantities which they contain. This is effected by making correspondent changes in both the members of the equation, so as to obtain, eventually, the unknown quantity on one side of the sign of equality, and known quantities on the other.

SIMPLE EQUATIONS.

Containing one unknown quantity.

To facilitate the solution of equations generally, the following principles or modes of operation must be observed.

Mode I.

52. Any quantity may be transposed, that is, removed from one member of the equation to the other, by changing its sign.*

EXAMPLES.

Given $7x+5a=6x+3b$ to find x.

By transposing $5a$ and $6x$, we have $7x—6x=3b—5a$, or $x=3b—5a$.

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* Hence, if any term be contained in both members of the equation with the same sign, it may be taken from both; and the signs of all the terms may be changed, without destroying the equation.

2. Given $8y-3=7y+5$ to find y. Result, $y=8$.
3. Given $15x-6=14x+20$, to find x. Result, $x=26$.
4. Given $20-13y=45-14y+4$, to find y.
Result, $y=29$.
5. Given $8x+15=15+9x-17$, to find x.
Result, $x=17$.
6. Given $25x-b=24x+2b-c$, to find x.
Result, $x=3b-c$.
7. Given $3b-9x=7b-8x-4d$, to find x.
Result, $x=4d-4b$.

Mode 2.

53. The co-efficient of an unknown quantity may be taken away, by dividing every term of the equation by it.

EXAMPLES.

Given $ax+ba=ad$, to find x.

By dividing by a, we have $x+b=d$; hence by transposition, $x=d-b$.

2. Given $5x=25$, to find x. Result, $x=5$.
3. Given $30x=180$, to find x. Result, $x=6$.
4. Given $6x+17=44-3x$, to find x. Result, $x=3$.
5. Given $3ax-4ab=2ax-6ac$, to find x.
Result, $x=4b-6c$.
6. Given $3x^2-10x=8x+x^2$, to find x.
Result, $x=9$.
7. Given $6y^3-8y^2=22y^2-3y^3$, to find y.
Result, $y=3\frac{4}{9}$

Mode 3.

54. An equation may be cleared of fractions by multiplying all its terms by any common multiple of the denominators.

EXAMPLES.

Given $\frac{x}{2}-\frac{x}{3}+\frac{x}{5}=1\frac{5}{6}$, to find x.

Multiply by 30, and then we have $15x-10x+6x=55$; whence $11x=55$, and $x=5$.

2. Given $\frac{x}{2}+\frac{x}{3}=\frac{x}{4}+7$, to find x. Result, $x=12$.

3. Given $\frac{x-5}{4}+6x=\frac{284-x}{5}$, to find x.

Result, $x=9$.

4. Given $x+\frac{11-x}{3}=\frac{19-x}{2}$, to find x. Result, $x=5$.

5. Given $3x+\frac{2x+6}{5}=5+\frac{11x-37}{2}$, to find x.

Result, $x=7$.

MODE 4.

55. When the unknown quantity is under a radical sign, such sign may be taken away, by first transposing the terms, so as to leave the radical quantity alone on one side of the equation, and then involving each member of the equation to the proper power.

EXAMPLES.

Given $\sqrt{x+40}=10-\sqrt{x}$, to find x.

Squaring both sides, $x+40=100-20\sqrt{x}+x$.

By transposition, $20\sqrt{x}=60$, or $\sqrt{x}=3$.

Squaring both members, $x=9$.

2. Given $\sqrt{x+1}-2=3$, to find x. Result, $x=24$.

3. Given $\sqrt[3]{3x+4}+3=6$, to find x. Result, $x=7\frac{2}{3}$.

4. Given $\sqrt{x-16}=8-\sqrt{x}$, to find x. Result, $x=25$.

5. Given $\sqrt{4a+x}=2\sqrt{b+x}-\sqrt{x}$, to find x.

Result, $x=\frac{b^2-2ab+a^2}{2a-b}$

6. Given $\sqrt{4a^3+x^2}=\sqrt[4]{4b^4+x^4}$, to find x.

Result, $x=\sqrt{\frac{b^4-4a^4}{2a^2}}$

7. Given $x+a+\sqrt{2ax+x^2}=b$ to find x.

Result, $x=\frac{b^2-2ab+a^2}{2b}$

Mode 5.

56. When the member of an equation, which contains the unknown quantity, is a complete square, cube, etc., the equation may be reduced by extracting the square root, cube root, etc. of both its members.*

EXAMPLES.

Given $x^3-6x^2+12x-8=343$, to find x.

Extracting the cube root, $x-2=7$, or $x=9$.

2. Given $x^2+4x+4=81$, to find x. Result, $x=7$.

3. Given $x^2+x+\frac{1}{4}=\frac{81}{4}$, to find x. Result, $x=4$.

4. Given $x^3+3ax^2+3a^2x+a^3=b^3$, to find x.

Result, $x=b-a$

5. Given $9x^2+17=98$, to find x. Result, $x=3$.

6. Given $x^4-4x^3+6x^2-4x+1=256$, to find x.

Result, $x=5$

7. Given $3x^5=96$, to find x. Result, $x=2$.

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* These modes of reducing equations depend upon a few self-evident truths. When two quantities are equal, and equal quantities are added to or subtracted from both, or both are divided or multiplied by the same number, or the same power or the same root of each is taken, the results in each case are equal to each other.

PROMISCUOUS EXAMPLES.

1. Given $\frac{1}{2}x-\frac{1}{4}x+\frac{1}{5}x=\frac{1}{2}$, to find x. Result, $x=1\frac{1}{9}$

2. Given $\sqrt{x}+\sqrt{2a+x}=\frac{4a}{\sqrt{2a+x}}$, to find x.

Result, $x=\frac{2}{3}a$

3. Given $\frac{a}{1+2x}+\frac{a}{1-2x}=2b$, to find x.

Result $x=\frac{1}{2}\sqrt{\frac{b-a}{b}}$

4. Given $a+x=\sqrt{a^2+x\sqrt{4b^2+x^2}}$, to find x.

Result, $x=\frac{b^2}{a}-a$

5. Given $2+\sqrt{3x}=\sqrt{4+5x}$, to find x.

Result, $x=12$.

6. Given $a-x=\frac{x^2}{a-x}$, to find x. Result, $x=\frac{a}{2}$

7. Given $\sqrt{a+x}+\sqrt{a-x}=\sqrt{ax}$, to find x.

Result, $x=\frac{4a^2}{a^2+4}$

8. Given $\frac{a}{1+x}+\frac{a}{1-x}=b$, to find x. Result, $x=\sqrt{\frac{b-2a}{b}}$

9. Given $\frac{7x+16}{21}-\frac{x+8}{4x-11}=\frac{x}{3}$, to find x.

Result, $x=8$.

10. Given $\frac{4x+3}{9}+\frac{7x-29}{5x-12}=\frac{8x+19}{18}$, to find x.

Result, $x=6$.

11. Given $\frac{5x+4}{2} : \frac{18-x}{5} :: 7 : 2$, to find x.*

Result, $x = 3\frac{5}{16}$

12. Given $\sqrt{5+x} + \sqrt{x} = \frac{15}{\sqrt{5+x}}$, to find x.

Result, $x = 4$.

13. Given $x^4 + 2x^2 + 1 = 4225$, to find x. Result, $x = 8$.

14. Given $x^6 - 9x^4 + 27x^2 - 27 = 2197$, to find x.

Result, $x = 4$.

15. Given $x^3 - 17 = 130 - 2x^2$, to find x. Result, $x = 7$.

16. Given $x^3 - 100 = 412 - 7x^3$, to find x. Result, $x = 4$.

17. Given $x^4 - 4x^3 + 6x^2 - 4x + 1 = 4096$, to find x.

Result, $x = 9$.

18. Given $x^3 - 6ax^2 + 12a^2x - 8a^3 = b^6$, to find x.

Result, $x = b^2 + 2a$.

19. Given $625x^4 - 1000x^3 + 600x^2 - 160x + 16 = 256x^4$, to find x. Result, $x = 2$.

20. Given $27a^3x^3 + 54ba^2x^2 + 36b^2ax + 8b^3 = 125c^3x^3$, to find x.

Result, $x = \frac{2b}{5c - 3a}$

21. Given $x + \sqrt{a-x} = \frac{a}{\sqrt{a-x}}$ to find x.

Result, $x = a - 1$.

22. Given $\sqrt{4 + \sqrt{x^4 - x^2}} = x - 2$, to find x.

Result, $x = 2\frac{1}{2}$

* If four numbers are proportional, the product of the extremes is equal to that of the means; hence an analogy is readily converted into an equation. Vide Article 62.

23. Given $\frac{4}{\sqrt{2+x}+\sqrt{x}}=\sqrt{2+x}$ to find x.

Result, $x=\frac{3}{4}$

SECTION V.

SIMPLE EQUATIONS.

Containing two unknown quantities.

RULE I.

57. Multiply or divide one or both equations, by such numbers or quantities as will make the co-efficients of one of the unknown quantities the same in both equations. Then take the sum or difference of these equations, according as the signs of the corresponding terms are different, or the same; the result will be a new equation, containing but one unknown quantity.* The value of the remaining unknown quantity may then be determined by the methods in the last section.

EXAMPLES.

Given $4x+y=34$, and $4y+x=16$, to find the values of x and y.

First, to eliminate x, we multiply the second equation by 4; whence, $16y+4x=64$. From this we take the first equation and $15y=30$; whence $y=2$.

The value of y being now obtained, we find $x=16-4y=16-8=8$.

Or to eliminate x, divide the first equation by 4, and we shall have $x+\frac{1}{4}y=\frac{34}{4}$, which subtracted from the second equation, leaves $\frac{15}{4}y=\frac{30}{4}$, whence $y=2$, as before.

---

*When, by this or any other process, a quantity or letter is caused to vanish from the equations, it is said to be *eliminated.*

2. Given $ax+by=m$, $cx-dy=n$, to determine the values of x and y.

To eliminate x, multiply the first equation by c, and the second by a, whence $acx+bcy=cm$, and $acx-ady=an$; by subtraction, $bcy+ady=cm-an$; and by dividing by $bc+ad$, $y=\frac{cm-an}{bc+ad}$.

Or, to eliminate y, multiplying the equations by d and b respectively, $adx+bdy=dm$, $bcx-bdy=bn$; adding these equations, $adx+bcx=dm+bn$; whence $x=\frac{dm+bn}{ad+bc}$.*

3. Given $\begin{cases} 5x+7y=50, \\ 8x+3y=39, \end{cases}$ to find the values of x and y.
Result, $x=3$, $y=5$.

4. Given $\begin{cases} 15x+34y=241, \\ 18x-17y=58, \end{cases}$ to find the value of x and y.
Result, $x=7$, $y=4$.

5. Given $\begin{cases} 6x-4y=34, \\ 9y-2x=27. \end{cases}$ Required the values of x and y.
Ans. $x=9$, $y=5$.

6. Given $\begin{cases} 11x+3y=100, \\ 4x-7y=4, \end{cases}$ to find the values of x and y.
Result, $x=8$, $y=4$.

7. Given $\begin{cases} \frac{x}{2}+\frac{y}{3}=7, \\ \\ \frac{x}{3}+\frac{y}{2}=8, \end{cases}$ to find x and y.
Result, $x=6$, $y=12$.

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*The known quantities being expressed in general terms, this solution is applicable to the following examples.
Thus if in the 6th example we make—

$a=11$, $b=3$, $m=100$, $c=4$, $d=7$, $n=4$, $x=\frac{700+12}{77+12}=8$,

$y=\frac{400-44}{77+12}=4$.

8. Given $\begin{cases} \frac{x}{15}-\frac{y}{20}=1. \\ \frac{y}{10}-\frac{x}{30}=1, \text{ to determine } x \text{ and } y. \end{cases}$

Result, $x=30$, $y=20$.

RULE 2.

58. Find, from each equation, the value of one unknown quantity in terms of the other quantities.

Assume the values thus obtained, as the different members of a new equation. The value of the unknown quantity contained in this equation may then be determined as in the last section.

EXAMPLES.

Given $\begin{cases} 7x+\frac{y}{7}=51, \\ 9x-\frac{y}{8}=61\frac{1}{2}, \text{ to find the values of } x \text{ and } y. \end{cases}$

From the first equation $y=357-49x$.
From the second equation $y=72x-490$,
Hence, $72x-490=357-49x$,
$72x+49x=357+490$,
$x=\frac{847}{121}=7$,
$y=357-49x=357-343=14$.

2. Given $\begin{cases} \frac{x}{7}+7y=99 \\ 7x+\frac{y}{7}=51. \end{cases}$ Required the values of x and y.

Ans. $x=7$, $y=14$.

3. Given $\begin{cases} \frac{x+2}{3}+8y=31, \\ \frac{y+5}{4}+10x=192. \end{cases}$ Required the values of x and y. Ans. $x=19$, $y=3$.

4. Given $\begin{cases} \frac{x}{5}+\frac{y}{9}=6 \\ \frac{x}{7}-\frac{y}{21}=2. \end{cases}$ Quere the values of x and y?

Ans. $x=20$, $y=18$.

5. Given $\begin{cases} \frac{2x}{3}+\frac{3y}{5}=22, \\ \frac{3x}{5}+\frac{2y}{3}=\frac{67}{3}. \end{cases}$ Quere the values of x and y?

Ans. $x=15$, $y=20$.

6. Given $\begin{cases} x-2y=d, \\ x:y::a:b. \end{cases}$ Quere the values of x and y?

Ans. $x=\frac{ad}{a-2b}$, $y=\frac{bd}{a-2b}$

7 Given $\begin{cases} \frac{x-2y}{4}+\frac{x}{5}=\frac{3x-y}{5}-5\frac{2}{5}, \\ \frac{x+y}{4}+\frac{x-y}{7}=8\frac{5}{7}. \end{cases}$ Quere the values of x and y? Ans. $x=20$, $y=8$.

RULE 3.

59. Find in either equation, the value of one unknown quantity, and substitute the value thus found in the other equation; whence will arise a new equation, containing but one unknown quantity, whose value may be determined as already taught.

EXAMPLES.

Given $\left\{\begin{array}{l} x+\frac{y}{2}=12 \\ 7x-3y=58 \end{array}\right\}$ to determine x and y.

From the first equation $x=12-\frac{y}{2}$. This value being substituted for x in the second equation, we have

$$84-\frac{7y}{2}-3y=58.$$

Multiplying by -2, $7y+6y-168=-116$.

Whence $y=\frac{168-116}{13}=4$,

$$x=12-\frac{y}{2}=12-2=10.$$

2. Given $\left\{\begin{array}{l} 5x-3y=150, \\ 10x+15y=825. \end{array}\right.$ Quere the values of x and y? Ans. $x=45$, $y=25$.

3. Given $\left\{\begin{array}{l} \frac{x}{3}-\frac{y}{5}=5, \\ \\ \frac{x}{5}+y=31, \text{ to find } x \text{ and } y. \end{array}\right.$

Result, $x=30$, $y=25$

4. Given $\left\{\begin{array}{l} x+y=24, \\ x:y::5:3, \text{ to find } x \text{ and } y. \end{array}\right.$

Result, $x=15$, $y=9$

5. Given $\left\{\begin{array}{l} \frac{x}{3}+3y=21. \\ \\ \frac{y}{3}+3x=29, \text{ to find } x \text{ and } y. \end{array}\right.$

Result, $x=9$, $y=6$.

Rule 4.

60. Multiply one of the equations by an indefinite quantity, and to the equation thus formed, add the other given equation.

Assume the sum of the co-efficients of one required quantity equal to 0, and thence determine the value of the assumed multiplier. The new equation will then contain but one unknown quantity, which may be found as before.

EXAMPLES.

Given $\left\{\begin{matrix} 7x-9y=59 \\ 3x+8y=132 \end{matrix}\right\}$ to find the values of x and y.

Multiply the second equation by the indefinite quantity m, and we shall have $3mx+8my=132m$.

Adding this equation to the first,

$$3mx+7x+8my-9y=132m+59.$$

Assume $3m+7=0$, whence $m=-\frac{7}{3}$,

$$\text{And } y=\frac{132m+59}{8m-9}=\frac{-308+59}{\frac{56}{-3}-9}=\frac{924-177}{56+27}=9.$$

$$\text{Hence } x=\frac{132-8y}{3}=\frac{132-72}{3}=\frac{60}{3}=20.$$

Or assuming $8m-9=0$, $m=\frac{9}{8}$,

$$\text{And } x=\frac{132m+59}{3m+7}=\frac{\frac{1188}{8}+59}{\frac{27}{8}+7}=\frac{1188+472}{27+56}=20.$$

2. Given $\begin{cases} 15x-17y=12, \\ 17x-10y=97, \end{cases}$ to find x and y.

Result, $x=11$, $y=9$.

3. Given $\begin{cases} 3x+7y=79, \\ 2y-\frac{1}{2}x=9, \end{cases}$ to find x and y.

Result, $x=10$, $y=7$.

4. Given $\begin{cases} 8x-5y=60, \\ 10y-3x=75, \end{cases}$ to determine x and y.

Result, $x=15$, $y=12$.

5. Given $\begin{cases} \frac{x}{2}+\frac{y}{4}=9 \\ \\ \frac{2}{5}x+\frac{3}{7}y=10\frac{6}{7}, \end{cases}$ to find x and y.

Result, $x=10$, $y=16$.

*Examples to exercise the foregoing rules.**

Given $\begin{cases} x-y=d, \\ xy=p, \end{cases}$ to find x and y.

Squaring the first equation, $x^2-2xy+y^2=d^2$.

Multiplying the second by 4, $4xy=4p$.

Taking the sum, $x^2+2xy+y^2=d^2+4p$.

Extracting the root, $x+y=\sqrt{d^2+4p}$, (A.)

Adding the first, $2x=d+\sqrt{d^2+4p}$, or $x=\frac{d+\sqrt{d^2+4p}}{2}$

Subtract the first equation from equation A, and

$$2y=\sqrt{d^2+4p}-d, \text{ or } y=\frac{\sqrt{d^2+4p}-d}{2}$$

* In the solution of these and other similar problems, the ingenious student will find expedients which are not clearly indicated in any of the preceding rules, by which his labor may be frequently abridged.

2. Given $\left\{\begin{array}{l} x+y=s, \\ xy=p, \end{array}\right.$ to find x and y.

Subtracting 4 times the second from the square of the first, we have $x^2-2xy+y^2=s^2-4p$.

Whence $x-y=\pm\sqrt{s^2-4p}$;* and $x=\frac{s\pm\sqrt{s^2-4p}}{2}$;

$y=\frac{s\mp\sqrt{s^2-4p}}{2}$.

3. Given $\left\{\begin{array}{l} x-y=m, \\ x^2-y^2=n, \end{array}\right.$ to find x and y.

Dividing the second by the first, $x+y=\frac{n}{m}$,

Hence, taking the half sum and half difference as before

$$x=\frac{n+m^2}{2m} \text{ and } y=\frac{n-m^2}{2m}.$$

4. Given $\left\{\begin{array}{l} x^2y-xy^2=a, \\ x^3-y^3=b, \end{array}\right.$ to find x and y.

Subtracting three times the first equation from the second, $x^3-3x^2y+3xy^2-y^3=b-3a$.

Hence, extracting the root, $x-y=\sqrt[3]{b-3a}$, which put $=c$, (A.)

Divide the first equation by this, whence $xy=\frac{a}{c}$ (B.)

From equations A and B, by proceeding as in the first example, we find

$$x=\tfrac{1}{2}\left\{\sqrt{c^2+\frac{4a}{c}}+c\right\}, y=\tfrac{1}{2}\left\{\sqrt{c^2+\frac{4a}{c}}-c\right\}$$

Or divide the sum of the given equations, by equation A and $x^2+2xy+y^2=\frac{a+b}{c}$,

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* As the square of a quantity whether positive or negative is always positive, the square root is often ambiguous; and therefore susceptible of the sign + or −. In this example, the ambiguity arises from the uncertainty which x or y is the greater.

Whence by extracting the square root, $x+y=\sqrt{\frac{a+b}{c}}$,

Then taking the half sum and half difference of this equation and equation A,

$$x=\tfrac{1}{2}\left\{\sqrt{\frac{a+b}{c}}+c\right\},\ y=\tfrac{1}{2}\left\{\sqrt{\frac{a+b}{c}}-c.\right\}$$

5. Given $\begin{cases} x+y=a, \\ x^3+y^3=b, \end{cases}$ to find x and y.

From the first $x^3+3x^2y+3xy^2+y^3=a^3$.

By subtraction, $3x^2y+3xy^2=a^3-b$.

Dividing by three times the first, $xy=\frac{a^3-b}{3a}$, which put $=c$.

From this and the first equation, we find as in example 2d.

$$x=\frac{a\pm\sqrt{a^2-4c}}{2}$$

$$y=\frac{a\pm\sqrt{a^2-4c}}{2}$$

Otherwise, dividing the second equation by the first,

$$x^2-xy+y^2=\frac{b}{a},$$

Hence, $4x^2-4xy+4y^2=\frac{4b}{a}$,

Squaring the first equation, $x^2+2xy+y^2=a^2$.

Subtracting the latter from the former, we have

$$3x^2-6xy+3y^2=\frac{4b-a^3}{a};$$

Consequently, $x^2-2xy+y^2=\frac{4b-a^3}{3a}$,

And by extracting the root, $x-y=\pm\sqrt{\frac{4b-a^3}{3a}}$,

Whence, $x=\frac{a\pm\sqrt{\frac{4b-a^3}{3a}}}{2}$ and $y=\frac{a\pm\sqrt{\frac{4b-a^3}{3a}}}{2}$

If we put $\frac{a^3-b}{3a}=c$, the values of x and y will be found the same as before.

6. Given $\begin{cases} x+y=s, \\ x-y=d, \end{cases}$ to find x and y in the terms of s and d.

Result, $x=\frac{s+d}{2}$, $y=\frac{s-d}{2}$

7. Given $\begin{cases} x+y=s, \\ x^2-y^2=d, \end{cases}$ to find as before.

$$x=\frac{s^2+d}{2s}, \quad y=\frac{s^2-d}{2s}$$

8. Given $\begin{cases} x:y::3:2, \\ x^2-y^2=20, \end{cases}$ to find x and y.

Result, $x=6$, $y=4$.

9. Given $\begin{cases} \frac{x-2}{5}-\frac{10-x}{3}=\frac{y-10}{4}, \\ \frac{2y+x}{3}-\frac{2x+y}{8}=\frac{x+17}{4}. \end{cases}$ Required the numerical values of x and y.

Ans. $x=7$, $y=10$.

10. Given $\begin{cases} x+1:y-1::5:2, \\ \frac{2x}{3}-\frac{5-y}{2}=\frac{41}{12}-\frac{2x-1}{4}, \end{cases}$ to find x and y.

Result, $x=4$, $y=3$.

11. Given $\begin{cases} \frac{y}{9}-\frac{4x-1}{18}=1-\frac{4+y}{3}+\frac{x-y}{6}, \\ x:3y::4:7, \end{cases}$ to find x and y.

Result, $x=12$, $y=7$.

12. Given $\begin{cases} \frac{4}{x}+\frac{5}{y}=\frac{9}{y}-1, \\ \frac{5}{x}+\frac{4}{y}=\frac{7}{x}+\frac{3}{2}, \end{cases}$ to find x and y.

Result, $x=4$, $y=2$.

13. Given $\begin{cases} x^2y+xy^2=120, \\ x^3+y^3=152. \end{cases}$ What are the values of x and y?

Ans. $x=.5$ or 3 . . $y=3$ or 5.

14. Given $\begin{cases} x+y=12, \\ x^3+y^3=468, \end{cases}$ to find x and y.

Result, $x=7$ or 5, $y=5$ or 7.

15. Given $\begin{cases} x^2+xy=112, \\ xy+y^2=84, \end{cases}$ to find x and y.

Result, $x=8$, $y=6$.

16.* Given $\begin{cases} ax+by=m, \\ dx+ey=n, \end{cases}$ to find x and y.

Result, $x=\dfrac{em-bn}{ae-db}$, $y=\dfrac{an-md}{ae-db}$

17. Given $\begin{cases} x^4-y^4=12240, \\ x^2+y^2=170, \end{cases}$ to find x and y.

Result, $x=11$, $y=7$.

18. Given $\begin{cases} x-y=2, \\ x^3-y^3=98, \end{cases}$ to determine x and y.

Result, $x=5$, $y=3$.

19. Given $\begin{cases} 7x+2y=30, \\ 5x+3y=34 \end{cases}$ to find x and y

Result, $x=2$, $y=8$.

20. Given $\begin{cases} \dfrac{1}{2}x+\dfrac{1}{3}y=8 \\ \dfrac{1}{3}x-\dfrac{1}{2}y=1 \end{cases}$ to find x and y.

Result, $x=12$, $y=6$.

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* When we have two equations, and two unknown quantities, neither of which rises above the first power, the required values may be obtained, from the results of the 16th example, by simple substitution.

21. Given $\begin{cases} \dfrac{2x+3y}{4}=5 \\ 2x=\dfrac{54-8y}{3} \end{cases}$ to find x and y.

Result, $x=1$, $y=6$.

Of arithmetical progression.

61. When the successive terms of a series of numbers are formed by the addition or subtraction of a constant number, the series is called an equidifferent one; or the numbers are said to be in arithmetical progression; and the constant number is termed the common difference. Thus 1, 3, 5, 7, 9. a, $a\pm d$, $a\pm 2d$, $a\pm 3d$, are series in arithmetical progression, the common differences being 2 and d, respectively.

In any series in arithmetical progression, it is obvious:

1. That the last term is equal to the first increased or diminished by the common difference multiplied by the number of terms less one.

2. That the sum of the extremes is equal to the sum of any other two which are equally distant from them, or to twice the mean when the number of terms is odd.

3. That the whole sum of such series is equal to the sum of the extremes multiplied by half the number of terms.

EXAMPLES.

1. What is the sum of a descending arithmetical series, of which the first term is 10, the common difference $\frac{1}{3}$, and the number of terms 21?

$$10-\frac{20}{3}=\frac{30}{3}-\frac{20}{3}=\frac{10}{3}=\text{last term.}$$

$$\left(\frac{10}{1}+\frac{10}{3}\right)\cdot\frac{21}{2}=\frac{40}{3}\times\frac{21}{2}=140=\text{sum of the series.}$$

2. Suppose a car descending an inclined plane moves 5 feet during the first second; 15 feet the second; 25 feet the

third; increasing the distance 10 feet each second of time; how far will it move in a minute?

Ans. 18000 feet.

3. Suppose 100 loads of wood placed in a straight line at intervals of 10 perches each, and that a wagoner is engaged to transport them to a place in the same line continued, 20 perches from the nearest pile; how far must he travel in performing the service, his journey being commenced at the place of deposit?

Ans. 160 *m.* 300 *p.*

4. What is the sum of the odd numbers 1, 3, 5, 7, etc., continued to n terms? Ans. n^2.

Of geometrical progression.

62. When a series of numbers, are such that any term multiplied by a constant number, either integral or fractional, produces the next following term, the numbers are said to be in geometrical progression, thus 2, 6, 18, 54; 216, 144, 96, 64; a, ar, ar^2, ar^3, ar^4, are in geometrical progression, the common multipliers being, respectively 3, $\frac{2}{3}$ and r.

The constant multiplier is usually termed the ratio. In these series we readily perceive that the last term is equal to the product of the first by the ratio involved to the power denoted by the number of terms less one:

Thus, the n term of the series a, ar, ar^2, etc. is ar^{n-1}.

To find the sum of a series in geometrical progression assume, the series $a+ar+ar^2 \quad ar^{n-2}+ar^{n-1}=s$, then multiplying by r, $ar+ar^2 \quad ar^{n-2}+ar^{n-1}+ar^n=rs$, whence by subtraction $-a+ar^n=$ or $(r^n-1)a=rs-s$ or $(r-1)s$, wherefore $\frac{r^n-1}{r-1}a=s$. If r is a proper fraction $\frac{1-r^n}{1-r}.a=s$.

EXAMPLES.

1. What is the sum of, $2+6+18$, etc. to 10 terms?

$$2.\left(\frac{3^{10}-1}{3-1}\right)=3^{10}-1=59048.$$

2. The first term of a geometrical series being a, the ratio, the proper fraction $\frac{m}{n}$, required the sum of the series continued to infinity.

As $\frac{m}{n}$ is a proper fraction, or n more than m, $\frac{m}{n}$ raised to an infinite power, may be considered $=0$.

Hence, the sum required $=\frac{a}{1-\frac{m}{n}}=\frac{na}{n-m}$.

3. Required the sum of $1+\frac{2}{3}+\frac{4}{9}+$ etc. to infinity.

Result, 3.

4. The series $1-\frac{1}{3}+\frac{1}{9}-\frac{1}{27}$, etc., being continued to infinity, required the amount?

Ans. $\frac{3}{4}$.

5. What is the sum of $1+\frac{1}{2}+\frac{1}{4}$, continued 20 terms?

Ans. $1\frac{524287}{524288}$.

Of harmonical progression.

63. If we have a series of numbers in arithmetical progression, their reciprocale, taken in the same order, form a series in harmonical progression. Thus, if $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}$ are in arithmetical progression, a, b, c, d are said to be in harmonical progression.

Let $\frac{1}{a}.\ \frac{1}{b}.\ \frac{1}{c}$ be in arithmetical progression, then Art. 61 $\frac{1}{a}+\frac{1}{c}=\frac{2}{b}$ $\therefore bc+ab=2ac$, or $(a+c)b=2ac$. From the former of these equations, $bc-ac=ac-ab$, whence $a:c::b-a:c-b$. Hence we have the relations of the terms, by which any two of three numbers in harmonical progression being given, the third may be found.

EXAMPLES.

1. The first and third terms of an harmonical series are 6 and 12, required the mean.

$$\frac{2\times 6\times 12}{6+12}=\frac{144}{18}=8.$$ The number sought.

2. The first and second terms of an harmonical series are 12 and 16; what is the third? Ans. 24.

SIMPLE EQUATIONS.

Containing three or more unknown quantities.

64. The unknown quantities may be eliminated one after another, by the first three rules laid down in the last section. Or multiply each given equation, except one, by an indefinite quantity; add the remaining given equation, and all the new equations together; and assume the sum of the co-efficients of each required quantity, except one, equal to 0; whence, as many equations as there are assumed multipliers will be formed; and thence the values of those multipliers may be determined, and consequently, the last unknown quantity.

There are other methods which may often be advantageously applied, but which can be learned only from practice.

EXAMPLES.

Given $\left\{\begin{array}{l} x+y+z=29, \\ x+2y+3z=62, \\ \frac{x}{2}+\frac{1}{3}y+\frac{1}{4}z=10. \end{array}\right\}$ to find the values of [x, y, and z.

Multiply the last equation by 2, $x+\frac{2}{3}y+\frac{1}{2}z=20$.

Subtract this from the first equation, $\frac{1}{3}y+\frac{1}{2}z=9$.

Multiply by 3, and $y+\frac{3}{2}z=27$.

Subtract the first equation from the second, and

$$y+2z=33. \quad (A.)$$

From this equation subtract the former, and we find

$$\frac{1}{2}z=6, \text{ or } z=12.$$

Take double the last equation from equation A, and $y=9$. Hence $y+z=21$, and this equation subtracted from the first leaves $x=8$.

Otherwise,—From the given equations,

$$x=29-y-z.$$

$$x=62-2y-3z.$$

$$x=20-\frac{2}{3}y-\frac{1}{2}z.$$

Whence $29-y-z=62-2y-3z$.

And $20-\frac{2}{3}y-\frac{1}{2}z=29-y-z$.

From the former of these, $y=33-2z$.

And from the latter, $y=27-\frac{3}{2}z$.

Therefore, $27-\frac{3}{2}z=33-2z$.

Or, $54-3z=66-4z$; $\therefore z=12$.*

But $y=33-2z=9$.

And $x=29-y-z=8$.

Or multiplying the second and third equations by m and n respectively, $mx+2my+3mz=62m$,

$$\frac{1}{2}nx+\frac{1}{3}ny+\frac{1}{4}nz=10n.$$

Hence, by adding these equations to the first,

$$x+mx+\frac{1}{2}nx+y+2my+\frac{1}{3}ny+z+3mz+\frac{1}{4}nz=29+62m+10n.$$

Assume $1+2m+\frac{1}{3}n=0$, and $1+3m+\frac{1}{4}n=0$

From the former of these equations $\frac{3}{2}+3m+\frac{1}{2}n=0$.

$$\therefore\frac{1}{2}+\frac{1}{4}n=0, \text{ or } n=-2.$$

Hence $m=-\frac{1}{6}$.

Whence, $x=\dfrac{29+62m+10n}{1+m+\frac{1}{2}n}=\dfrac{29-\frac{62}{6}-20}{1-\frac{1}{6}-1}=8.$

Again making $1+m+\frac{1}{2}n=0$, and $1+3m+\frac{1}{4}n=0$,

We have $n=-\frac{8}{5}$, and $m=-\frac{1}{5}$.

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* The sign $\therefore$ is used in place of the word therefore.

Consequently, $y=\dfrac{29+62m+10n}{1+2m+\frac{1}{3}n}=9$,

Hence, $z=12$.

2. Given $\begin{cases} x+y+z=53, \\ x+2y+3z=105, \\ x+3y+4z=134, \text{ to find } x, y, \text{ and } z. \end{cases}$

Subtract the first from the second, $y+2z=52$.
Second from third, $y+z=29$.
Subtracting the last, $z=23$.
$\therefore y=6$.
And $x=24$.

3. Given $\begin{cases} x+y=20, \\ x+z=24, \\ y+z=30, \text{ to find } x, y, \text{ and } z. \end{cases}$
Result, $x=7$, $y=13$, $z=17$.

4. Given $\begin{cases} 7x+5y+2z=79, \\ 8x+7y+9z=122, \\ x+4y+5z=55, \text{ to determine the values of} \end{cases}$
x, y, and z. Result, $x=4$, $y=9$, $z=3$.

5. Given $\begin{cases} x-y=2, \\ x-z=3, \\ y+z=9, \text{ to find } x, y, \text{ and } z. \end{cases}$
Result, $x=7$, $y=5$, $z=4$.

6. Given $\begin{cases} 2x+3y+4z=38, \\ 5x+7y+11z=98, \\ 7x+9y+15z=132. \text{ to find } x, y, \text{ and } z. \end{cases}$
Result, $x=3$, $y=4$, $z=5$

7. Given $\begin{cases} ax+by+cz=m, \\ dx+ey+fz=n, \\ gx+hy+kz=p, \text{ to find } x, y, \text{ and } z. \end{cases}$

Result,* $\begin{cases} x=\dfrac{kem-hfm+hcn-bkn+bfp-ecp}{aek-ahf+dhc-dbk+gbf-gec} \\ y=\dfrac{akn-afp+dcp-dkm+gfm-gcn}{aek-ahf+dhc-dbk+gbf-gec} \\ z=\dfrac{aep-ahn+dhm-dbp+gbn-gem}{aek-ahf+dhc-dbk+gbf-gec} \end{cases}$

8. Given $\begin{cases} v+x+y=30, \\ v+x+z=31, \\ v+y+z=32, \\ x+y+z,=33, \text{ to find } v, x, y, \text{ and } z. \end{cases}$

Result, $v=9$, $x=10$, $y=11$, $z=12$.

9. Given $\begin{cases} \dfrac{4x+3y+z}{10}-\dfrac{2y+2z-x+1}{15}=5+\dfrac{x-z-5}{5}, \\ \dfrac{9x+5y-2z}{12}-\dfrac{2x+y-3z}{4}=\dfrac{7y+z+3}{11}+\dfrac{1}{6}, \\ \dfrac{5y+3z}{4}-\dfrac{2x+3y-z}{12}+2z=y-1+\dfrac{3x+2y+7}{6} \end{cases}$

Required the values of x, y, and z.

Ans. $x=9$, $y=7$, $z=3$.

10. Given $\begin{cases} 3v+4x+5y+6z=102, \\ 4v-6x+4y-3z=6, \\ 2v+3x-5y+4z=11, \\ 6v+5x-7y+z=7, \text{ to find } v, x, y, \text{ and } z. \end{cases}$

Result, $v=5$, $x=4$, $y=7$, $z=6$.

Examples producing simple equations.

Required to find two numbers whose difference shall be 4, and the difference of the squares 64.

Let x = the greater number, and y = the less.

Then, by the question, $\begin{cases} x-y=4, \\ x^2-y^2=64. \end{cases}$

Dividing the 2d equation by the 1st, $x+y=16$

* From these general results, the values of x, y, z, in the preceding examples, may be found by simple substitution.

Taking the sum and difference of this equation and the first, $2x=20$, and $2y=12$.

$$\therefore x=10, \text{ and } y=6.$$

Otherwise. Let x = the less number, then $x+4$ = the greater:

Whence, by the question, $\overline{x+4}\,]^2-x^2=64$,

$$\text{Or } 8x+16=64.$$

$\therefore$ by transposition and division, $x=6$, and $x+4=10$.

2. A person having bought three loads of grain, the first, consisting of 30 bushels of rye, 20 of barley, and 10 of wheat, for 230 francs; the second, containing 15 bushels of rye, 6 of barley, and 12 of wheat, for 138 francs; the third consisting of 10 bushels of rye, 5 of barley, and 4 of wheat, for 75 francs; required the price per bushel of each of these kinds of grain.

Suppose $\left\{\begin{array}{l}\text{a bushel of rye cost } x,\\ \text{a bushel of barley, } y,\\ \text{and a bushel of wheat, } z,\end{array}\right\}$ francs

Then by the question, $\left\{\begin{array}{l}30x+20y+10z=230, (A.)\\ 15x+6y+12z=138,\\ 10x+5y+4z=75.\end{array}\right.$

Multiplying the 2d equation by 2, and the 3d by 3,

$$30x+12y+24z=276,$$

$$30x+15y+12z=225, (B.)$$

Taking the difference of these equations,

$$3y-12z=-51. (C.)$$

Take the difference of equations $(A.)$ and $(B.)$ and

$$5y-2z=5.$$

Multiplying this by 6, $30y-12z=30$.

And subtracting equation $(C.)$

$$27y=81. \quad \text{Whence, } y=3.$$

$$\therefore z=\frac{15-5}{2}=5, \text{ and } x=\frac{23-6-5}{3}=4.$$

3. Required to find two numbers, such that their difference multiplied by their product, shall be 308, and the difference of their cubes 988.

Let x = the greater, and y = the less.

Then from the question,

$$(x-y).xy, \text{ or } x^2y-xy^2=308.$$
$$\text{And } x^3-y^3=988.$$

From the latter taking thrice the former,

$$\text{We have } x^3-3x^2y+3xy^2-y^3=64.$$

And taking the cube root, $x-y=4$. $(A.)$

Dividing the sum of the two primary equations by this last,

$$\frac{x^3+x^2y-xy^2-y^3}{x-y}=x^2+2xy+y^2=324.$$

Hence, by extracting the square root, $x+y=18$. $(B.)$

Taking the half sum and half difference of equations $(A.)$ and $(B.)$

$$x=\frac{18+4}{2}=11, \text{ and } y=\frac{18-4}{2}=7.$$

4. The sum of two numbers multiplied by their difference is 20, and the difference of their 4th powers 1040; what are the numbers?

Let x = the greater, y = the less.

Then from the *data*, $\begin{cases} \overline{x+y}.\overline{x-y}=x^2-y^2=20. \\ \text{And } x^4-y^4=1040. \end{cases}$

Dividing the latter by the former, $x^2+y^2=52$.

Taking the sum and difference of the first and last equations, $2x^2=72$, and $2y^2=32$.

$$\therefore x=\sqrt{36}=6.$$
$$\text{And } y=\sqrt{16}=4.$$

5. The difference of two numbers is 5, and the difference of their cubes 1685; what are the numbers?

Let x = the greater, y = the less.

Then, *per data*, $\begin{cases} x-y=5, \\ x^3-y^3=1685. \end{cases}$

By division, $\frac{x^3-y^3}{x-y}=x^2+xy+y^2=337$, (*A.*)

Squaring the first equation, $x^2-2xy+y^2=25$.

By subtraction, $3xy=312$.

Adding $\frac{1}{3}$ of this equation to equation (*A.*)

$$x^2+2xy+y^2=441.$$

Extracting the square root, $x+y=21$.

From this and the first equation, $x=13$, and $y=8$.

Or from the second equation subtract the cube of the first, whence $3x^2y-3xy^2=1560$; and dividing this equation by three times the first, $xy=104$; whence as in page 63,

$$x=\frac{\sqrt{25+416}+5}{2}=13. \text{ and } y=\frac{\sqrt{25+416}-5}{2}=8.$$

6. Required to find two numbers, such that half the first added to a third of the second shall make 9, and a fourth of the first, added to a fifth of the second, shall make 5

Ans. 8 and 15.

7. A testator bequeaths to his widow $\frac{2}{5}$ of his estate, to his son $\frac{1}{3}$, and the remainder, which is found to be 756 dollars, to his daughter; what was the estate left?

Ans. 2835 dollars.

8. There are two numbers in the ratio of 3 to 5, and their sum is one fourth of the difference of their squares; what are the numbers? Ans. 6 and 10.

9. The ages of a man and his wife, at the time of marriage, were in the ratio of 7 to 6, and at the end of 30 years, they are found to be as 17 to 16. Quere their ages at the former period? Ans. 21 and 18.

10. A post being $\frac{1}{4}$ of its length in the earth, $\frac{1}{3}$ in the water, and 10 feet above the water; what was its whole length? Ans. 24 feet.

11. Divide 1100 dollars among A. B. and C., so that B. shall have 100 dollars more than A., and C. 150 more than B.; what is the share of each?

Ans. A. 250, B. 350, C. 500.

12. There are 21 persons, whose ages form an equi-different series; the age of the eldest is 4 times that of the youngest, and the sum of all their ages is 525 years; how old is the youngest? Ans. 10 years.

13. At a certain election, 1296 persons voted, and the successful candidate had a majority of 120; how many voted for each?

Ans. 708 for one, and 588 for the other.

14. A servant agreed to serve his master for £8 a year and his livery, but was turned away at the end of 7 months, and received only £2 13*s.* 4*d.* besides has livery; what is the price of his livery? Ans. £4 16*s.*

15. A servant having eloped from his master, travels 14 hours in the day, at the rate of 3½ miles an hour; at the end of two days, a courier is sent in pursuit, who rides 9 hours in the day, at the rate of 7 miles an hour; in what time, and at what distance will he overtake him?

Ans. 7 days, and 441 miles.

16. The sun's mean daily motion in the ecliptic is 59′ 8″, and that of the moon 13° 10′ 35″; what is the time of a mean synodic revolution of the moon, viz. a revolution from conjunction with the sun to conjunction again?

Ans. 29 days, 12 hours, 44 minutes nearly.

17. A farmer, having hired a laborer, on condition, that for every day he wrought he should receive 50 cents, and for every day he was idle he should forfeit 20 cents, finds at the end of 420 days, that neither is indebted to the other; how many days did he labor. Ans. 120.

18. A certain number, consisting of two places of figures, is equal to the difference of the squares of its digits; and if 36 be added to it, the order of the digits will be inverted; what is the number? Ans. 48.

19. A vintner proposing to mix three sorts of wine, viz.

at 65 cents per gallon, at 45 cents, and at 35 cents; the number of gallons at 65 cents, to the number at 35 cents, being as 5 to 3, so as to compose a hogshead worth 50 cents per gallon. Quere the number of each?

Ans. 22½ at 65, 27 at 45, and 13½ at 35.

20. There are 4 equi-different numbers, whose sum is 56, and the sum of whose squares is 864; what are the numbers?

Ans. 8, 12, 16, and 20.

21. If 38 federal dollars and 11 English crowns be given for 271 French francs, and 76 dollars and 33 crowns for 610 francs; at how many francs were the dollar and crown respectively estimated?

Ans. the dollar $5\frac{13}{38}$, the crown $6\frac{2}{11}$.

22. There is a number consisting of three digits in arithmetical progression, whose sum is 21; and if to the number 396 be added, the sum will be expressed by the same digits in an inverted order; what is the number? Ans. 579.

23. Required to find four numbers, such that the continued product of the 1st, 2d, and 3d shall be 24; the product of the 1st, 2d, and 4th, 30; of the 1st, 3d, and 4th, 40; and of the 2d, 3d, and 4th, 60. Result, 2, 3, 4, 5.

24. It is required to find three numbers, such that the first multiplied by the sum of the other two shall be 96; the second multiplied by the sum of the other two shall be 105; and the third multiplied by the sum of the former two shall be 117. The numbers are 6, 7, and 9.

25. What fraction is that, to the numerator of which, if 1 be added, the value will be $\frac{1}{6}$, but if to the denominator 1 be added, the value will be $\frac{1}{4}$? Ans. $\frac{4}{15}$.

26. A person bought a chaise, horse, and harness, for 150 dollars; the price of the chaise was twice the price of the harness, and the price of the horse as much and half as much as the price of the chaise and harness; what was the cost of each?

Ans. harness 20 dollars, chaise 40, horse 90.

27. Two persons, A. and B., have the same income; A. saves $\frac{1}{5}$ of his yearly; but B., by spending as much in 3

years as A. does in 4, finds himself, at the end of five years, 200 dollars in debt; what was the income?

Ans. 600 dollars.

28. A. and B. put equal sums in trade: A. gained a sum equal to $\frac{1}{4}$ of his stock, and B. lost 450 dollars; when A.'s money was double of B's; what was the sum laid out by each? Ans. 1200 dollars.

29. Two persons comparing their ages, find them, at present, in the ratio of 7 to 5, but that 30 years ago, they were in the ratio of 2 to 1; what are their ages?

Ans. 70 and 50.

30. A. and B. began trade with equal sums of money. In the first year A. gained £40, and B. lost £40; but in the second, A. lost one third of what he then had, and B. gained a sum which was £40 less than twice what A. had lost; when it appeared that B. had twice as much as A.; what sum had each of them at first? Ans. £320.

31. A farmer sold 96 loads of hay to two persons. To the first one half, and to the second one fourth of what his stack contained; how many loads were in the stack?

Ans. 128.

32. If 116 be divided into four parts, in such manner, that the first being increased by 5, the second diminished by 4, the third multiplied by 3, and the fourth divided by 2, the results will all be the same; what are the parts?

Ans. 22, 31, 9, and 54.

33. A shepherd, in time of war, was plundered by a party, who took from him $\frac{1}{4}$ of his flock, and $\frac{1}{4}$ of a sheep more; another party took from him $\frac{1}{3}$ of what he had left, and $\frac{1}{3}$ of a sheep more; afterward, a third party took $\frac{1}{2}$ of what remained, and $\frac{1}{2}$ a sheep more, when he had but 25 left; how many had he at first? Ans. 103.

34. A person has two horses and a gig; the gig is worth 150 dollars. When the first horse is attached to the gig, the value of the two is twice that of the second; but when the second horse is put to the gig, the value is three times

that of the first horse; what were the horses respectively worth? Ans. the first 90 dollars, the second 120.

35. When a company at a tavern came to pay their reckoning, they found, that if there had been three persons more, they would have had a shilling a piece less to pay; but if there had been two less, they would have had a shilling a piece more to pay; required the number of persons, and the quota of each?

Ans. 12 persons; quota of each 5s.

36. There are three equi-different numbers, whose sum is 324, and the first is to the third as 5 to 7; what are the numbers? Ans. 90, 108, and 126.

37. A man and his wife usually drank a cask of beer in 12 days; but when the man was from home, it lasted the woman 30 days; how long would the man alone be in drinking it? Ans. 20 days.

38. A. and B. can perform a piece of work in 8 days, A. and C. in 9 days, and B. and C. in 10 days; how many days would they severally require to perform the same work? Ans. A. $14\frac{4}{9}$ days, B. $17\frac{1}{7}$, and C. $23\frac{1}{7}$.

39. The hypothenuse of a right angled triangle is 13, and the area 30; what are the other two sides?*

Ans. 12 and 5.

40. There are four numbers in geometrical progression, the sum of the extremes is 18, and the sum of the means 12; what are the numbers?† Ans. 2, 4, 8, and 16.

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* To solve this problem, it must be recollected that the square of the hypothenuse is equal to the sum of the squares of the other two sides, and that the area is half the product of those sides.

† To solve this question by simple equations, put x, y to denote the means, then since, when three numbers are in geometrical progression, the square of the mean is equal to the product of the extremes, the extremes will be expressed by $\frac{x^2}{y}$, and $\frac{y^2}{x}$. Hence $x+y=12$, and $\frac{x^2}{y}+\frac{y^2}{x}=18$. From the last $x^3+y^3=18xy$. Subtracting this equation from the cube of the first, $3x^2y+3xy^2=1728-18xy$. But $3x^2y+3xy^2=3(x+y).\ xy=36xy$. Hence $54xy=1728$ or $xy=32$. Whence x and y are found as in page 64.

41. Required to find four numbers in geometrical progression, such that the difference of the means shall be 100, and the difference of the extremes 620.

The numbers are 5, 25, 125, and 625.

42. There are three numbers in harmonical progression, whose sum is 26, and the difference between the second, and third exceeds the difference between the first and second by 2; what are the numbers? Ans. 6, 8, and 12.

43. A person having £27 6*s.* sterling in guineas and crowns, pays a debt of £14 17*s.*, and then finds the number of guineas left equal to the number of crowns paid away; and the number of crowns left equal to the number of guineas paid; how many of each had he at first?

Ans. 21 of each.

44. There is a number consisting of two digits, which is equal to four times the sum of those digits; but 18 being added, the order of digits will be inverted. Quere the number? Ans. 24.

45. Required to find three such numbers, that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third added together shall make 46; $\frac{1}{3}$ of the first, $\frac{1}{4}$ of the second, and $\frac{1}{5}$ of the third shall make 35; and $\frac{1}{4}$ of the first, $\frac{1}{5}$ of the second, and $\frac{1}{6}$ of the third shall make $28\frac{1}{2}$. Result, 12, 60, and 80.

46. A person having £22 14*s.* sterling in crowns, guineas and moidores, finds that if he had as many guineas as he has crowns, and as many crowns as he has guineas, the whole sum would be £36 6*s.*; but if he had as many moidores as he has crowns, and as many crowns as he has moidores, his money would amount to £45 16*s.* How many had he of each?

Ans. 26 crowns, 9 guineas, and 5 moidores.

47. Required two numbers, such that their sum, difference, and product may be as the numbers 3, 2 and 5 respectively. Result, 10 and 2.

48. The sum of the first and third terms of four numbers in geometrical progression is 74, and the sum of the second and fourth 444; what are the numbers?

Ans. 2, 12, 72, and 432.

£1. 7*s.* rated of a moidore

49. A. and B. enter into trade with different sums, A. gains 750 dollars, and B. loses 250 dollars; when their stocks are found to be as 3 to 2. But if A. had lost 250 dollars, and B. had gained 500 dollars, their stocks would have been as 5 to 9. What was the original stock of each?

Ans. A. 1500 dollars, B. 1750 dollars.

SECTION VII.

QUADRATIC EQUATIONS.

65. When an equation contains the square and simple power of an unknown quantity; or in general, two powers, one of whose indices is double the other, of an unknown quantity, whether that unknown quantity be simple or compound, it is called an *adfected quadratic equation.*

RULE I.

Arrange the terms, by transposition or otherwise, so that the highest power of the unknown quantity shall be contained in the first, and the other power in the second term on the left side of the equation; and the term or terms consisting of known quantities, on the other side.

When the highest power of the unknown quantity contains a co-efficient, divide the equation by that co-efficient. Then add, to each member of the equation, the square of half the co-efficient of the unknown quantity, (or the numerical part of the second term,) and the first member of the equation will be a complete square.

Extract the root and find the value of the unknown quantity by the former methods.

RULE 2.

66. Arrange the terms as before. When the highest power of the unknown quantity has a co-efficient, multiply the equation by four times that co-efficient, and to each

member of the resulting equation, add the square of the coefficient of the unknown quantity contained in the primary equation; then the first member of the resulting equation will be a complete square.

Extract the root, and proceed by the former methods, to determine the value of the quantity sought.

EXAMPLES.

Given $2x^2-9x=266$, to find the value of x.

By rule 1.—Dividing by 2, $x^2-\frac{9}{2}x=133$.

Adding $\left(\frac{9}{4}\right)^2$ to each member, $x^2-\frac{9}{2}x+\frac{81}{16}=\frac{2209}{16}$

Extracting the root, $x-\frac{9}{4}=\frac{47}{4}$

$$\therefore x=14.$$

By rule 2.—Multiplying by 8, $16x^2-72x=2128$.
Adding 81 to each member, $16x^2-72x+81=2209$
Extracting the square root, $4x-9=47$.

$$\therefore x=\frac{47+9}{4}=14, \text{ as before.}$$

2. Given $ax^2+bx=c$, to find the value of x.

By rule 1.—Dividing by a, $x^2+\frac{b}{a}x=\frac{c}{a}$.

Adding $\frac{b^2}{4a^2}$ to each, $x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=\frac{b^2+4ac}{4a^2}$

By evolution, $x+\frac{b}{2a}=\frac{\sqrt{b^2+4ac}}{2a}$

By transposition, $x=\frac{\sqrt{b^2+4ac}-b}{2a}$

By rule 2.—Multiplying by $4a$, $4a^2x^2+4abx=4ac$.
Adding b^2 to each, $4a^2x^2+4abx+b^2=4ac+b^2$.

By evolution, $2ax+b=\sqrt{4ac+b^2}$.

By transposition and division, $x=\dfrac{\sqrt{4ac+b^2}-b}{2a}$

3. Given $2x^{\frac{2}{3}}+3x^{\frac{1}{3}}=2$, to find the value of x.

By rule 1. $x^{\frac{2}{3}}+\frac{3}{2}x^{\frac{1}{3}}=1$.

Comp. the sqr. $x^{\frac{2}{3}}+\frac{3}{2}x^{\frac{1}{3}}+\frac{9}{16}=\frac{25}{16}$

By evolution, $x^{\frac{1}{3}}+\frac{3}{4}=\frac{5}{4}$

$$\therefore x^{\frac{1}{3}}=\frac{5-3}{4}=\frac{1}{2}$$

Whence, $x=\left(\frac{1}{2}\right)^3=\frac{1}{8}$

By rule 2. $16\,x^{\frac{2}{3}}+24x^{\frac{1}{3}}=16$.

Adding 9 to each, $16\,x^{\frac{2}{3}}+24x^{\frac{1}{3}}+9=25$.

By evolution, $4\,x^{\frac{1}{3}}+3=5$.

By transp. and division, $x^{\frac{1}{3}}=\frac{1}{2}$

Therefore. $x=\frac{1}{8}$

4. Given $x^4-12x^3+44x^2-48x=9009$, to determine x.

This equation is equivalent to to the following:

$$x^4-12x^3+36x^2+8x^2-48x=9009,$$

Or to this, $(x^2-6x)^2+8.(x^2-6x)=9009$.

In which, x^2-6x may be considered as the unknown quantity.

Hence, proceeding as in the former examples,

$$(x^2-6x)^2+8.(x^2-6x)+16=9025.$$

$$\therefore x^2-6x+4=95, \text{ or } x^2-6x=91.$$

Whence, $x^2-6x+9=100.$

$$\therefore x-3=10, \text{ or } x=13.$$

5. Given $\sqrt{(x-\frac{1}{x})}+\sqrt{(1-\frac{1}{x})}=x$, to find the value of x.

Multiplying by $\sqrt{x}$, $\sqrt{x^2-1}+\sqrt{x-1}=x\sqrt{x}$.

Squaring this, $x^2-1+2\sqrt{x^3-x^2-x+1}+x-1=x^3$.

By transposing and changing the signs,

$$x^3-x^2-x+1-2\sqrt{x^3-x^2-x+1}=-1.$$

Here $\sqrt{x^3-x^2-x+1}$, may be considered as the unknown quantity.

Hence, adding 1 to each side,

$$x^3-x^2-x+1-2\sqrt{x^3-x^2-x+1}+1=0.$$

$$\therefore \sqrt{x^3-x^3-x+1}-1=0, \text{ or } \sqrt{x^3-x^3-x+1}=1.$$

$$\therefore x^3-x^2-x+1=1;$$

Whence $x^2-x=1.$

$$x^2-x+\frac{1}{4}=\frac{5}{4}. \therefore x-\frac{1}{2}=\sqrt{\frac{5}{4}}=\frac{1}{2}\sqrt{5}.$$

$$\therefore x=\frac{1}{2}+\frac{1}{2}\sqrt{5}.$$

6. Given $3x^{2n}-2x^n+3=11$, to find x.

$$3x^{2n}-2x^n=8.$$

By rule 1. $x^{2n}-\frac{2}{3}x^n=\frac{8}{3}$

$$x^{2n}-\frac{2}{3}x^n+\frac{1}{9}=\frac{25}{9}$$

$$x^n-\frac{1}{3}=\frac{5}{3}, \text{ or } x=2^{\frac{1}{n}}.$$

By rule 2. $36x^{2n}-24x^n=96.$

$36x^{2n}-24x^n+4=100.$

$6x^n-2=10.$

$x^n=\frac{12}{6}=2$, or $x=\sqrt[n]{2}$, as before.

7. Given $x^{4n}-2x^{3n}+x^n=132$, to find the value of x

This equation may be changed to

$$x^{4n}-2x^{3n}+x^{2n}-x^{2n}+x^n=132.$$

Which is $(x^{2n}-x^n)^2-(x^{2n}-x^n)=132.$

Complete the square,

$$(x^{2n}-x^n)^2-(x^{2n}-x^n)+\frac{1}{4}=132\tfrac{1}{4}=\frac{529}{4}$$

Extract the root, $x^{2n}-x^n-\frac{1}{2}=\frac{23}{2}$

$$\therefore x^{2n}-x^n=12.$$

Complete the square, $x^{2n}-x^n+\frac{1}{4}=12\tfrac{1}{4}=\frac{49}{4}$

Extract the root, $x^n-\frac{1}{2}=\frac{7}{2}$

$$\therefore x^n=4, \text{ or } x=\sqrt[n]{4}.$$

8. Given $x^2+12x=64$. Required the value of x.

Ans $x=4$.

9. Given $7x^2+5x+4=82$, to find x. Ans. $x=3$.

10. Given $\frac{1}{2}x^2-\frac{1}{3}x+7\frac{2}{3}=8$, to find x.

Ans. $x=1\frac{1}{2}$

11. Given $x^3+20x^3=224$, to find x. Ans. $x=2$.

12. Given $5x^2-4x+3=159$, to find x. Ans. $x=6$.

13. Given $6x+\frac{35-3x}{x}=44$, to find x.

Ans. $x=7$, or $\frac{5}{6}$.*

* This example falls under the ambiguous case, which is explained in the next note.

14. Given $4x-\frac{14-x}{x+1}=14$, to find the value of x.

Ans. $x=4$.

15. Given $3x-\frac{1121-4x}{x}=2$, to find the value of x.

Ans. $x=19$.

16. Given $5x-\frac{3x-3}{x-3}=2x+\frac{3x-6}{2}$, to find x.

Ans. $x=4$.

67. The foregoing solutions have been effected by executing the whole process in each case; but it is more convenient in practice, as well as more elegant, to solve all such equations by a few general *formulæ*. In order to which it is to be observed, that all adfected quadratic equations are reducible to one of the following forms.

$$\left.\begin{array}{l} 1.\ x^2+2ax=b. \\ 2.\ x^2-2ax=b. \\ 3.\ x^2-2ax=-b. \end{array}\right\}(A.)$$

These equations, when the squares are completed, become,

$$\begin{array}{l} 1.\ x^2+2ax+a^2=a^2+b \\ 2.\ x^2-2ax+a^2=a^2+b \\ 3.\ x^2-2ax+a^2=a^2-b. \end{array}$$

Hence, by evolution and transposition, we have

$$\begin{array}{l} 1.\ x=-a+\sqrt{a^2+b}. \\ 2.\ x=a+\sqrt{a^2+b}. \\ 3.\ x=a\pm\sqrt{a^2-b}.* \end{array}$$

Adfected quadratic equations are sometimes expressed by one of the following forms.

$$\left.\begin{array}{l} 1.\ ax^2+bx=c. \\ 2.\ ax^2-bx=c. \\ 3.\ ax^2-bx=-c. \end{array}\right\}(B.)$$

See note, page 90.

Multiplying these equations severally by $4a$, and adding b^2 to each member, they become

1. $4a^2x^2+4abx+b^2=b^2+4ac.$
2. $4a^2x^2-4abx+b^2=b^2+4ac.$
3. $4a^2x^2-4abx+b^2=b^2-4ac.$

The roots being extracted, the values of x are easily found.

1. $x=\dfrac{-b+\sqrt{b^2+4ac}}{2a}$

2. $x=\dfrac{b+\sqrt{b^2+4ac}}{2a}$

3. $x=\dfrac{b\pm\sqrt{b^2-4ac}}{2a}$ *

From these *formulæ*, any adfected quadratic equation may be resolved, by assuming for a and b, in the equations $(A,)$ and for a, b, and c, in the equations $(B,)$ such values as will render the general equation identical with the particular one under consideration, and substituting the values thus assumed in the final equation.

Simple quadratic equations are evidently solvable by the same *formulæ* if we assume the co-efficient in the second term $=0$.

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*The value of x, in the third case, is ambiguous; because, $\sqrt{x^2-2ax+a^2}=x-a$, or $a-x$, and $\sqrt{4a^2x^2-4abx+b^2}=2ax-b$, or $b-2ax$; and there is nothing in the general equation to show whether x is greater or less than a in the former case, or than $\dfrac{b}{2a}$ in the latter. The ambiguity does not exist in the second case, because $\sqrt{a^2+b}$ must be greater than a, and $\sqrt{b^2+4ac}$ greater than b, and therefore, cannot be equal respectively to $a-x$, and $b-2ax$, unless x be negative. When the sign of x is doubtful, the first and second cases are also ambiguous.

17. Given $9x^2-7x=116$.

Dividing by 9, $x^2-\frac{7}{9}x=\frac{116}{9}$

Comparing this equation with the 2d form, (A,) we have

$$2a=\frac{7}{9} \text{ and } b=\frac{116}{9},$$

Hence, $x=\frac{7}{18}+\sqrt{(\frac{49}{324}+\frac{116}{9})}=\frac{72}{18}=4.$

Or comparing the given equation with the 2d form, (B.)

$$a=9,\ b=7,\ c=116.$$

Hence, $x=\frac{7+\sqrt{49+4176}}{18}=\frac{7+65}{18}=4.$

18. Given $\sqrt{10+x}-\sqrt[4]{10+x}=2$, to find the value of x.

Considering $\sqrt[4]{10+x}$ as the quantity sought, and comparing the equation with the 2d form, (A,) we have

$$2a=1,\ b=2.$$

$$\therefore \sqrt[4]{10+x}=\frac{1}{2}+\sqrt{2\tfrac{1}{4}}=2.$$

$$10+x=2^4=16, \text{ and therefore, } x=6.$$

$$\text{Or } \sqrt{10+x}-2=\sqrt[4]{10+x}.$$

Hence by involution, $10+x-4\sqrt{10+x}+4=\sqrt{10+x}$.

Hence, $14+x=5\sqrt{10+x}$.

By involution, $196+28x+x^2=250+25x$.

Hence, $x^2+3x=54$.

Comparing this with the 1st form, (A,)

$$2a=3, \text{ and } b=54.$$

$$\therefore x=-\frac{3}{2}+\sqrt{(54\tfrac{9}{4})}=\frac{12}{2}=6.$$

19. Given $x+\sqrt{5x+10}=8$, to determine the value of x.

Multiplying by 5, and adding 10 to each member,

$$5x+10+5\sqrt{5x+10}=50.$$

Considering $\sqrt{5x+10}$ as the quantity sought; and comparing the equation with the first form, (A,) we have

$$2a=5, \text{ and } b=50.$$

Hence $\sqrt{5x+10}=-\frac{5}{2}+\sqrt{(50\frac{25}{4})}=5.$

By involution and transposition, $5x=25-10.$

Therefore, $x=3.$

Otherwise.—By transposition, $\sqrt{5x+10}=8-x.$

By involution, $5x+10=64-16x+x^2,$

Or $x^2-21x=-54.$

Comparing this with the 3d form, (B,)

$$a=1,\ b=21, \text{ and } c=54.$$

Hence, $x=\frac{21\pm\sqrt{441-216}}{2}=18$ or $3.$*

20. Given $\begin{cases} x^2y^2+xy=1260, \\ x+y=12, \text{ to find } x \text{ and } y. \end{cases}$

Comparing the 1st equation with form 1, (A.)

$$a=\tfrac{1}{2},\ b=1260.$$

Hence, $xy=-\frac{1}{2}+\sqrt{(\frac{1}{4}+1260)}=35.$

If in this equation, we substitute for y, its value, $12-x$, found from the 2nd we have

$$12x-x^2=35 \text{ or } x^2-12x=-35.$$

Comparing this with third form (A)

$$a=6, \text{ and } b=35.$$

Hence, $x=6\pm1=7$ or $5.$

One of which being taken for x, the other will be the value of y.

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* The ambiguity is *introduced* into this solution by the involution of the given equation; for $(8-x)^2=(x-8)^2$; and one of the values of x corresponds to the supposition, that $5x+10=(x-8)^2$, and the other to the true one, that $5x+10=(8-x)^2$.

21. Given $\left\{\begin{matrix}\frac{3x^2}{y^2}-\frac{5x}{y}=2\\ xy=8\end{matrix}\right\}$ to find the values of x and y.

Comparing the 1st equation with form 2d, $(B,)$

$$a=3,\ b=5,\ c=2.$$

Hence, $\frac{x}{y}=\frac{5+\sqrt{25+24}}{6}=2.$

$$\therefore\ x=2y, \text{ and } 2y^2=8.$$

$$\therefore\ y=\sqrt{\frac{8}{2}}=2, \text{ and } x=4.$$

22. Given $\left\{\begin{matrix}x^2+xy=12,\\ xy-2y^2=1,\end{matrix}\right\}$ to find x and y.

Assume $vy=x$, then $v^2y^2=x^2$, and $vy^2=xy$.

$$\therefore\ v^2y^2+vy^2=12, \text{ or } y^2=\frac{12}{v^2+v}$$

$$vy^2-2y^2=1, \text{ or } y^2=\frac{1}{v-2}$$

Hence, $\frac{12}{v^2+v}=\frac{1}{v-2}$, or $12v-24=v^2+v$.

Therefore, $v^2-11v=-24$.

Comparing this with form 3d, $(A,)$ $a=\frac{11}{2}, b=24$.

Whence, $v=\frac{11}{2}\pm\sqrt{(\frac{121}{4}-24)}=8$ or 3.

$$y=\sqrt{\frac{1}{v-2}}=\frac{1}{\sqrt{6}} \text{ or } 1,\ x=vy=\frac{8}{\sqrt{6}} \text{ or } 3.$$

Otherwise.—Comparing the first equation with 12 times the second, $x^2+xy=12xy-24y^2$,

$$\therefore\ x^2-11xy=-24y^2.$$

Hence, considering y as a given quantity.

By form 3. $(B.)$ $x=\frac{11\pm5}{2}y=8y$ or $3y$.

Whence, $y=\frac{1}{6}\sqrt{6}$ or 1, and $x=\frac{4}{3}\sqrt{6}$ or 3.

23. Given $\left\{\begin{array}{l}\frac{x^2}{y}+\frac{y^2}{x}=28\\ x+y=12.\end{array}\right\}$ to find x and y.

Assume $x=vy$,

Then $v^2y+\frac{y}{v}=28=a.$

$$vy+y=12=b.$$

$$\therefore \frac{a}{v^2+\frac{1}{v}}=y=\frac{b}{v+1},$$

Multiplying by v^3+1.

$$av=b\{v^2-v+1\}$$

$$bv^2-(b+a)v=-b$$

By form 3. (B_3) $v=\dfrac{b+a\pm\sqrt{a^2+2ab-3b^2}}{2b}=3$ or $\frac{1}{3}$.

Hence, $y=3$ or 9.

Wherefore $x=9$ or 3.

24. Given $\left\{\begin{array}{l}\frac{x^2}{y}+x+y+\frac{y^2}{x}=a.\\ \frac{x^4}{y^2}+x^2+y^2+\frac{y^4}{x^2}=b.\end{array}\right\}$

Assume $x+y=s$, $xy=p$.

Then $\frac{x^2}{y}+\frac{y^2}{x}=a-s.$

And multiplying by $xy=p$, $x^3+y^3=ap-sp$.

But, $x^3+y^3=s^3-3sp$.*

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* $x+y=s$, $xy=p$.

From the square of the first equation subtract twice the second,

$$x^2+y^2=s^2-2p.$$

From this equation subtract the second, and

$$x^2-xy+y^2=s^2-3p.$$

Multiply by the first, and $x^3+y^3=s^3-3sp$.

$$\therefore ap-sp=s^3-3sp, \text{ and } p=\frac{s^3}{a+2s}(C,)$$

Again, $x^2+y^2=s^2-2p.$

Also, $\frac{x^4}{y^2}+\frac{y^4}{x^2}=(a-s)^2-2p=a^2-2as+s^2-2p.$

Hence, $\frac{x^4}{y^2}+x^2+y^2+\frac{y^4}{x^2}=a^2-2as+2s^2-4p=b.$

By substitution, $(C,)$ $a^2-2as+2s^2-\frac{4s^3}{a+2s}=b.$

Clearing the equation of fractions, $a^3-2as^2=ab+2bs.$

$$\therefore s^2+\frac{b}{a}s=\frac{a^2-b}{2}$$

Consequently, $s=\frac{-b}{2a}+\sqrt{(\frac{b^2}{4a^2}+\frac{a^2-b}{2})}$

Whence, $p=\frac{s^3}{a+2s}$ becomes known.

Now, $(x-y)^2=(x+y)^2-4xy=s^2-4p.$

Or $x-y=\pm\sqrt{s^2-4p}.$

$$\therefore x=\frac{s\pm\sqrt{s^2-4p}}{2}, \text{ and } y=\frac{s\mp\sqrt{s^2-4p}}{2}$$

Questions producing quadratic equations.

1. Required to find two numbers, such that their sum added to their product shall be 47; and the sum of the squares shall be 74.

Let x and y denote the numbers.

Then $x+y+xy=47$,
And $x^2+y^2=74$, } by the data.

Add twice the first equation to the second, and

$$x^2+2xy+y^2+2x+2y=168.$$

Or, $(x+y)^2+2.(x+y)=168.$

Hence form 1, $(A.)$ $x+y=-1+13=12.$

This equation subtracted from the first, leaves

$$xy=35.$$

By subtracting twice the last equation from the 2d,

$$x^2-2xy+y^2=4$$

Hence by evolution, $x-y=\pm 2$.

$$\therefore x=7 \text{ or } 5.$$

2. A. and B. sold 130 ells of silk, for 42 crowns; 40 ells belonged to A. and 90 to B. Now, A. sold for a crown $\frac{1}{3}$ of an ell more than B. did. How many ells did each sell for a crown?

Let x = the number B. sold,
Then $x+\frac{1}{3}$ = the number A. sold, } for a crown,

$$\therefore \frac{90}{x}=\text{number of crowns received for B.'s silk.}$$

And $\frac{40}{x+\frac{1}{3}}=\frac{120}{3x+1}=$number received for A.'s silk.

$$\therefore \frac{90}{x}+\frac{120}{3x+1}=42.$$

$$270x+90+120x=126x^2+42x.$$

By transposition and division, $21x^2-58x=15$.

Comparing this with form 2, $(B,)$ we find

$$a=21,\ b=58,\ c=15.$$

Hence $x=\frac{\sqrt{1260+3364}+58}{42}=\frac{126}{42}=3$, number of ells B. sold for a crown.

$\therefore 3\frac{1}{3}$, number A. sold for a crown.

3. There are two square yards, paved with stone, each stone being a foot square; the side of one yard exceeds that of the other by 12 feet, and the number of stones in the two is 2120; what are the sides?

Ans. 26 and 38 feet.

4. A laborer dug two trenches, one of which was 6 yards longer than the other, for £17 16*s*., and the digging of each cost as many shillings per yard as there were yards in its length; what was the length of each?

Ans. 10 and 16 yards.

5. A. and B. set out from two towns, which were distant from each other 247 miles, and traveled till they met. A. traveled 9 miles a day, and the number of miles walked by B. in a day, increased by 3, was equal to the number of days occupied by the journey. Required the number of days they were traveling, and the number of miles passed over by each?

Ans. 13 days; and A. went 117 miles, and B. 130.

6. A. and B. having bought 41 oxen, for which each of them paid 420 dollars, find A.'s worth a dollar a head more than B.'s; how must they divide them?

Ans. A. 20, B. 21.

7. Divide the number 14 into two parts, whose product shall be 48. Ans. the numbers are 6 and 8.

8. Given the sum of two numbers=9, and the sum of the squares=45; what are the numbers?

Ans. 6 and 3.

9. What two numbers are those, whose sum, product, and the difference of their squares, are equal to each other?

Ans. $\frac{3}{2}+\frac{1}{2}\sqrt{5}$, and $\frac{1}{2}+\frac{1}{2}\sqrt{5}$.

10. There are four numbers in arithmetical progression, of which the product of the two extremes is 22, and that of the means 40; what are the numbers?

Ans. 2, 5, 8, and 11.

11. There are three numbers in geometrical progression, whose sum is 7, and the sum of their squares 21; what are the numbers? Ans. 1, 2, and 4.

12. Required to find two numbers, such that the less may be to the greater, as the greater is to 12; and the sum of the squares may be 45. Ans. 3 and 6.

13. What two numbers are those, whose difference is 2, and the difference of their cubes 98? Ans. 3 and 5.

14. What two numbers are those, whose sum is 6, and the sum of whose cubes is 72? Ans. 2 and 4.

15. Required to find two such numbers, that their product shall be 20, and the difference of their cubes 61? Ans. 4 and 5.

16. Required to divide the number 5 into two such parts, that if each part be divided by the other, the sum of the quotient shall be $2\frac{1}{6}$. Ans. 3 and 2.

17. Divide 12 into two parts, so that their product may be equal to 8 times their difference. Ans. 8 and 4.

18. There are two numbers, the sum of whose squares is 89, and their sum, multiplied by the greater, is 104; what are the numbers? Ans. $\frac{1}{2}\sqrt{2}$, and $\frac{3}{2}\sqrt{2}$, or 8 and 5.

19. What number is that, which being divided by the product of its two digits, the quotient is $5\frac{1}{2}$; but when 9 is subtracted from it, the remainder is expressed by the same digits in an inverted order? Ans. 32.

20. Required to divide 13 into three such parts, that their squares may be equi-different, and the sum of those squares may be 75. Ans. 1, 5, 7.

21. The sum of three equi-different numbers is 12, and the sum of their 4th powers 962; what are the numbers? Ans. 3, 4, and 5.

22. There are three equi-different numbers, such that the square of the least, added to the product of the other two, makes 28; but the square of the greatest, added to the product of the other two, makes 44; what are the numbers? Ans. 2, 4 6.

23. Three merchants, A. B. C., on comparing their gains, finds they amount, collectively, to 1,444 dollars; that B.'s gain added to the square root of A.'s makes 920 dollars; and that B.'s gain added to the square root of C.'s makes 912 dollars; how much did they severally gain? Ans. A. 400, B. 900, C. 144.

24. What two numbers are those, whose sum, added to the square of the sum, makes 702, and whose difference, subtracted from the square of the difference, leaves 56?

Ans. 17 and 9.

25. The sum of two numbers is 10, and the sum of their 4th powers 1552; what are the numbers?*

Ans. 6 and 4.

26. There are four numbers in arithmetical progression, the common difference of which is 4, and their continued product 9945; what are the numbers?

Ans. 5, 9, 13, 17.

27. The sum of three numbers in harmonical proportion is 191, and the product of the first and last is 4032; what are the numbers? Ans. 56, 63, 72.

28. The sum of two numbers added to their product makes 31; but the sum subtracted from the sum of the squares, leaves 48. Quere the numbers?

Ans. 7 and 3.

29. Required to find three numbers in continued proportion, whose sum shall be 26, and the sum of their squares 364? Ans. 2, 6, and 18.

30. Required to find two numbers whose product shall be 320, and the difference of their cubes to the cube of their difference as 61 to 1? Ans. 20 and 16.

31. If 700 dollars be divided among four persons, so that their shares may be in geometrical progression, and the difference of the extremes to the difference of the means as 37 to 12, what will be the several shares?

Ans. 108, 144, 192, and 256 dollars.

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* In examples of this nature, we may assume letters to denote the half sum and half difference of the required numbers; whence, expressions for the numbers themselves are readily obtained, and these being involved, and the powers added together, the odd powers of the unknown quantity will disappear. Hence, if the unknown quantity does not rise higher than the 5th power, the solution may be effected by qnadratic equations.

32. The sum of two numbers is 11, and the sum of their 5th powers 17831; quere the numbers?

Ans. 7 and 4.

33. The difference of two numbers is 8, and the difference of their 4th powers is 14560; required the numbers?*

Ans. 11 and 3.

34. What number is that, from the square of which, if 9 be subtracted, and the remainder be multiplied by the number itself, the product will be 80? Ans. 5.

35. Required a number, from the square of which, 30 being deducted, and the remainder multiplied by the number itself, the product shall be 56? Ans. $2+3\sqrt{2}$.

36. The continued product of five equi-different numbers is 945, and their sum 25; what are the numbers?

Ans. 1, 3, 5, 7, and 9.

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*In this example, assuming for the half sum and half difference, we obtain a cubic equation, whence it appears that questions of this nature are not generally solvable by quadratics. When, however, we have an equation of the form $x^3+ax=b$; if $b=mn$, and $m^2+a=n$, the equation may be reduced to a quadratic. For multiplying by x, and adding m^2x^2 to each member, the given equation becomes $x^4+ax^2+m^2x^3=m^2x^2+bx$, or $x^4+nx^2=m^2x^2+nmx$, whence by completing the squares and extracting the roots, $x^2+\frac{n}{2}=mx+\frac{n}{2}$, or $x=m$.

In the above example, x being made $=$ half the sum of the required numbers, we find $x^3+16x=455=7\times 65$; where $7^2+16=65$, whence, $x=7$. If $x^3-ax=mn$, and $m^2-a=n$; or if $x^3-ax=-mn$, and $a-n^2=u$, the equation may be reduced to a quadratic, and x found $=m$, as before. But if $x^3-ax=mn$, and $a-m^2=n$, or $x^3-ax=-mn$, and $m^2-a=n$; the equation may be changed to a quadratic, from which a second quadratic $x^2-mx=\pm n$, will arise, whence $x=\frac{m\pm\sqrt{m^2\pm 4n}}{2}$.

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37. The sum of three numbers in geometrical progression is 35, and the mean is to the difference of the extremes as 2 to 3; what are the numbers? Ans. 5, 10, and 20.

38. The sum of three numbers in geometrical progression is 13, and the product of the mean by the sum of the extremes is 30; required the numbers? Result, 1, 3, 9.

39. There is a number consisting of three digits in geometrical progression; the number itself is to the sum of its digits as 124 to 7; and if 594 be added to it, the order of the digits will be inverted; what is the number?*

Ans. 248.

Promiscuous examples to exercise the foregoing rules.

1. What number is that, to the double of which, if 44 be added, the sum will be equal to 4 times the number proposed? Ans. 22.

2. A gentleman, meeting 4 poor persons, divided a dollar among them in such manner that their several shares composed an equi-different series; and the sum given to the last was 4 times that given to the first. Quere their several shares? Ans. 10, 20, 30, and 40 cents.

3. A sum of money being divided among 6 poor persons, the second received 10*d.*, the third 14*d.*, the fourth

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* Adfected quadratic equations may be reduced to simple quadratics, and solved without completing the square, in the following manner:

If the highest power of the unknown quantity has a co-efficient, divide the whole equation by it; then assume the unknown quantity, equal to another unknown quautity with half the co-efficient united by the opposite sign. Substitute this new value, and a simple quadratic will arise. Thus, if $x^2+2ax=b$, assume $z-a=x$, then $z^2-2az+a^2=x^2$, $2az-2a^2=2ax$, hence $z^2-a^2=x^2+2ax=b$, and $z=\sqrt{b+a^2}$, hence $x=z-a=-a+\sqrt{b+a^2}$.

In like manner, the second term of an equation of a higher order may be taken away by assuming a new unknown quantity with $\frac{1}{3}$, $\frac{1}{4}$, etc., of the co-efficient of the next term united by the opposite sign, in place of the quantity sought.

25*d.*, the fifth 28*d.*, and the sixth 33*d.* less than the first. The whole sum divided was 10*d.* more than three times what the first received. How much did they severally receive? Ans. 40*d.*, 30*d.*, 26*d.*, 15*d.*, 12*d.*, and 7*d.*

4. A mercer, having cut 19 yards from each of 3 equal pieces of silk, and 17 yards from another of the same length, finds the four remnants together measure 142 yards; what was the original length of each piece?

Ans. 54.

5. A grazier, having two flocks of sheep, containing the same number, sells 39 from one, and 93 from the other, and then finds one flock twice as numerous as the other; what number did each of them contain at first?

Ans. 147.

6. From each of 16 coins an artist filed the value off 20 cents, when the coins, being examined, were found worth only 11 dollars 68 cents; what was the original value of each? Ans. 93 cents.

7. The hold of a ship, containing 442 gallons, is emptied in 12 minutes by two buckets; the greater of which, holding twice as much as the less, is emptied twice in three minutes, and the less is emptied three times in two minutes. Quere the number of gallons held by each bucket?

Ans. 26 and 13.

8. A trader maintained himself for three years at an annual expense of £50, and in each of those years augmented that part of his stock which was not expended by $\frac{1}{2}$ thereof. At the end of the third year, his original stock was doubled; what was that stock? Ans. £740.

9. A gentleman having a rectangular yard, 100 feet by 80, purposes to make a gravel walk of equal width half round it, so as to occupy one fourth of the ground. What must be the width of the walk? Ans. 11.8975 feet.

10. Required to find a fraction, to the numerator of which, if 4 be added, the value will be $\frac{1}{2}$; but if 7 be added to the denominator, the value will be $\frac{1}{5}$?

Ans. $\frac{5}{18}$.

11. What two numbers are those, whose difference, sum, and product are as the numbers 2, 3, and 5 respectively?

Ans. 10 and 2.

12. A vintner, having mixed a quantity of brandy and water, finds that if he had mixed 6 gallons more of each, he would have had 7 gallons of brandy for every 6 of water; but if he had mixed 6 gallons less of each, he would have had 6 gallons of brandy for every 5 of water. Quere the number of gallons of each?

Ans. 78 of brandy, and 66 of water.

13. A. and B. together, are able to perform a piece of work in 15 days; after working jointly 6 days, B. finishes it alone in 30 days; in what time would each of them singly effect it? Ans. A. in $21\frac{3}{7}$ days, B. in 50 days

14. A company of smugglers found a cave which would exactly hold their cargo, viz: 13 bales of cotton, and 33 casks of rum; but while they were unloading, a revenue cutter appeared, on which they sailed away with 9 casks and 5 bales, having filled $\frac{2}{3}$ of the cave; how many bales, or how many casks would the cave contain?

Ans. 24 bales, or 72 casks.

15. There are two numbers, whose sum is to their difference as 8 to 1, and the difference of whose squares is 128; what are the numbers? Ans. 18 and 14.

16. Required those two numbers, whose sum is to the less as 5 to 2; and whose difference multiplied by the difference of their squares is 135? Ans. 9 and 6.

17. A merchant laid out a certain sum upon a speculation, and found, at the end of a year, that he had gained £69. This he added to his stock, and at the end of the second year found he had gained as much per cent. as in the first. Continuing in this manner, and each year adding to his stock the gain of the preceding, he found, at the end of the fourth year, that his stock was to the sum first laid out as 81 to 16. Quere the sum first invested?

Ans. £138.

18. There are two numbers, whose sum is to the greater as 40 is to the less, and whose sum is to the less as 90 to the greater; what are the numbers? Ans. 36 and 24.

19. The area of a rectangular parallelogram is 960 yards, and the length exceeds the breadth by 16 yards; what are the sides? Ans. 40 and 24.

20. The area of a rectangular parallelogram is 480, and the sum of the length and breadth 52. Quere the sides? Ans. 40 and 12.

21. The sides of a right angled triangle form an equidifferent series, whose common difference is 3; what are the sides? Ans. 15, 12, and 9.

22. There are three numbers, the difference of whose differences is 5, their sum is 20, and their continued product 130; required the numbers? Ans. 2, 5, and 13.

23. The sum of three numbers is 21, the sum of the squares of the greatest and least is 137, and the difference of the differences is 3. Quere the numbers? Ans. 4, 6, and 11.

24. There is a number consisting of two digits, which, divided by the sum of its digits, has a quotient greater by 2 than the first digit. But, the digits being inverted, and divided by the sum of the digits increased by unity, the quotient is equal to the first digit increased by 4. Quere the number? Ans. 24.

25. Required to find four numbers in geometrical progression, whose sum shall be 15, and the sum of the squares 85?* Ans. 1, 2, 4, and 8.

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*Note.—If we assume x,y to denote the means, $\frac{x^2}{y}$, and $\frac{y^2}{x}$ will represent the extremes; the problem will then be the same as example 24, page 94. The following solution includes an expedient which may be sometimes used with advantage.

Let x denote the first term, and y the common multiplier.

Then $x+xy+xy^2+xy^3=15=a$; $x^2+x^2y^2+x^2y^4+x^2y^6=85=b$.

26. There are five numbers in geometrical progression, whose sum is 242, and the sum of their squares 29524; what are the numbers? Ans. 2, 6, 18, 54, 162.

27. There are six numbers in geometrical progression, the sum of the extremes is 99, and the sum of the other four terms is 90. Quere the numbers?

Ans. 3, 6, 12, 24, 48, and 96.

Then $x^2=\frac{a^2}{(y^3+y^2+y+1)^2}=\frac{b}{y^6+y^4+^2+1}$

But from the nature of progressionals,

$$y^3+y^2+y+1=\frac{y^4-1}{y-1};\text{ and } y^6+y^4+y^2+1=\frac{y^8-1}{y^2-1};$$

The above equation may therefore be expressed thus,

$$\frac{a^2(y-1)(y-1)}{(y^4-1)(y^4-1)}=\frac{b.(y^2-1)}{y^8-1}=\frac{b(y-1)(y+1)}{(y^4-1)(y^4+1)}$$

$$\therefore \frac{a^2(y-1)}{y^4-1}=\frac{b(y+1)}{y^4+1}$$

And dividing the numerator and denominator of the first member by $(y-1)$, and clearing the equations of fractions,

$$\therefore a^2(y^4+1)=b(y^4+2y^3+2y^2+2y+1.)$$
$$\therefore (a^2-b)y^4-2by^3-2by^2-2by+(a^2-b)=0.$$

$$\therefore (a^2-b)y^2-2by-2b-2b.\frac{1}{y}+(a^2-b)\frac{1}{y^2}=0.$$

Now $(y+\frac{1}{y})^2=y^2+2+\frac{1}{y^2}$

$$\therefore (a^2-b)(y+\frac{1}{y})^2=(a^2-b)y^2+2(a^2-b)+(a^2-b)\frac{1}{y^2}$$

Hence, $(a^2-b)(y+\frac{1}{y})^2-2b(y+\frac{1}{y})=2a^2$, a quadratic equation.

$$\therefore 14(y+\frac{1}{y})^2-17(y+\frac{1}{y}=45 \quad \therefore y+\frac{1}{y}=\frac{17+53}{28}=\frac{5}{2}.$$

$2y^2-5y=-2$, $y=\frac{5\pm3}{4}=2$ or $\frac{1}{2}$.

Hence 1, 2, 4, 8, or 8, 4, 2, 1, are the numbers sought.

The 26th and 27th examples may be solved in a similar way.

28. The American dollar consists of 1485 parts by weight of pure silver, and 179 of copper. Quere the specific gravity of this dollar, the specific gravity of pure silver being 11092, and that of copper 9000?

Ans. 10821.

29. A person having set out from a certain place, travels one mile the first day, two the second, etc., in arithmetical progression. In six days another sets out from the same place, and travels in the same direction at the rate of 15 miles a day. In how many days will they come together?

Ans. In 9, and in 20 days after the first sets out.

30. Heavy bodies near the earth's surface are known to fall $16\frac{1}{12}$ feet in the first second of time; and to pass over spaces, which, reckoned from the commencement of the fall, are as the squares of the time. Now, a hail stone, being observed to descend 595 feet during the last second; quere how long was it falling, and from what height did it descend? Ans. 19 seconds, and 5806 feet.

31. The arithmetical mean of two numbers exceeds the geometrical mean by 13; and the geometrical mean exceeds the harmonical mean by 12. What are the numbers? Ans. 234, and 104.

32. The fore wheel of a carriage makes six revolutions more than the hind wheel in going 120 yards; but if the circumference of each wheel be increased one yard, it will make only four revolutions more than the hind wheel in going the same distance. Required the circumference of each? Ans. The fore wheel 4 yards, the hind 5.

Section VIII.

ON RATIOS.

68. Ratio is the relation that two quantities of the same kind bear to each other.

Ratio is estimated by the quotient arising from the division of the first term by the second. Thus, if $\frac{a}{b}=\frac{c}{d}$, a is said to have the same ratio to b that c has to d; or the quantities a, b, c, d, are termed *proportionals;* which is briefly expressed thus, $a : b :: c : d$.

The first and third terms are called *antecedents*, the second and fourth *consequents.**

69. When four quantities are proportionals, the product of the first and fourth is equal to the product of the second and third; and reciprocally.

If $\frac{a}{b}=\frac{c}{d}$, by multiplying by bd, $ad=bc$.

Reciprocally, if $ad=bc$, $\frac{ad}{bd}=\frac{bc}{bd}$, or $\frac{a}{b}=\frac{c}{d}$.

70. When four quantities are proportionals, the sum of the first and second is to the second, as the sum of the third and fourth is to the fourth.

$$\text{If } \frac{a}{b}=\frac{c}{d};\ \frac{a}{b}+1=\frac{c}{d}+1, \text{ or } \frac{a+b}{b}=\frac{c+d}{d},$$

$$\text{Or, } a+b : b :: c+d : d.$$

If the word difference be substituted for sum, the proposition will still be a true one.

$$\text{For } \frac{a}{b}=\frac{c}{d},\ \therefore \frac{a}{b}\sim 1=\frac{c}{d}\sim 1, \text{ or } \frac{a\sim b}{b}=\frac{c\sim d}{d},$$

$$\text{Or, } a\sim b : b :: c\sim d : d.$$

71. If four quantities are proportionals, the sum of the first and second is to their difference as the sum of the third and fourth is to their difference.

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*The terms are similarly designated when more then four are concerned.

For by the last article, $\frac{a+b}{b}=\frac{c+d}{d}$, and $\frac{a\sim b}{b}=\frac{c\sim d}{d}$,

And dividing the former by the latter,

$\frac{a+b}{a\sim b}=\frac{c+d}{c\sim d}$, or $a+b : a\sim b :: c+d : c\sim d$.

72. When any number of quantities are proportionals, as one antecedent is to its consequent, so is the sum of all the antecedents to the sum of all the consequents.

Let $a : b :: c : d :: e : f :: g : h$, etc., then (art. 69.)

$ad=bc$, $af=be$, $ah=bg$, etc., also $ab=ba$.

$\therefore ab+ad+af+ah$, etc. $=ba+bc+be+bg$, etc.

Or, $a\times\{b+d+f+h$, etc.$\}=b\times\{a+c+e+g$, etc.$\}$

$\therefore$ (art. 69,) $a : b :: a+c+e+g$, etc. $: b+d+f+h$, etc.

73. When four quantities are proportionals, if the first and second be multiplied or divided by any quantity, and likewise the third and fourth, the resulting quantities will be proportionals.

If $a : b :: c : d$, $\frac{a}{b}=\frac{c}{d}$, but $\frac{a}{b}=\frac{ma}{mb}$;

And $\frac{c}{d}=\frac{nc}{nd}$, $\therefore \frac{ma}{mb}=\frac{nc}{nd}$, or $ma : mb :: nc : nd$.

The demonstration is manifestly applicable when m and n are, one or both, fractional numbers.

74. When the first and third of four proportionals are multiplied or divided by any number, and also the second and fourth, the resulting quantities are proportionals.

For $\frac{a}{b}=\frac{c}{d}$ $\therefore \frac{ma}{b}=\frac{mc}{d}$ and $\frac{ma}{nb}=\frac{mc}{nd}$,

Or $ma : nb :: mc : nd$, m and n being any numbers, either integral or fractional.

75. If four quantities are proportionals, the like powers or roots of these quantities will be proportionals.

If $a : b :: c : d$, then $\frac{a}{b}=\frac{c}{d}$, and $\frac{a^n}{b^n}=\frac{c^n}{d^n}$,

Also, $\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}=\frac{c^{\frac{1}{n}}}{d^{\frac{1}{n}}}$, or $a^n : b^n :: c^n : d^n$.

And $a^{\frac{1}{n}} : b^{\frac{1}{n}} :: c^{\frac{1}{n}} : d^{\frac{1}{n}}$.

76. A ratio compounded of several ratios is indicated by the continued product of the quantities which denote the component ratios. Thus, the ratio compounded of the ratios of $a : b$, of $c : d$, and of $e : f$, is indicated by

$$\frac{a}{b}\times\frac{c}{d}\times\frac{e}{f}, \text{ or } \frac{ace}{bdf}.$$

The ratio which the first of a series of quantities, of like kind, has to the last, is the same as the ratio compounded of the ratios of the first to the second, of the second to the third, etc. to the last.

Let a, b, c, d, e, f, be quantities of a like kind, then

$$\frac{a}{f}=\frac{a}{b}\times\frac{b}{c}\times\frac{c}{d}\times\frac{d}{e}\times\frac{e}{f},$$

77. In two ranks of proportionals, if the corresponding terms be multiplied together, the products will be proportionals.

Let $a : b :: c : d$, and $e : f :: g : h$.

Then, $\frac{a}{b}=\frac{c}{d}$, $\frac{e}{f}=\frac{g}{h}$ $\therefore \frac{ae}{bf}=\frac{cg}{dh}$,

Or, $ae, : bf : : cg : dh$.

The demonstration may be easily applied to any number of proportions.

78. A ratio compounded of two, three, four, etc., equal ratios, is called the duplicate, triplicate, quadruplicate, etc. of one of the component ratios.

The ratio compounded of any number of equal ratios is the same as the ratio of such power of the first term, as is indicated by the number of component ratios to a like power of the second.

Let $a : b :: b : c$, then $\frac{a}{b}=\frac{b}{c}$

$\frac{a^2}{b^2}=\frac{ab}{bc}=\frac{a}{c}$, or, $a^2 : b^2 :: a : c$.

Again, let $a : b :: b : c :: c : d :: d : e$, etc.

Then, $\frac{a}{b}=\frac{a}{b}$, $\frac{a}{b}=\frac{b}{c}$, $\frac{a}{b}=\frac{c}{d}$, $\frac{a}{b}=\frac{d}{e}$, $\frac{a}{b}=\frac{e}{f}$, etc. to n, equations; and multiplying the first and second members respectively together,

$$\frac{a^n}{b^n}=\frac{a}{b}\times\frac{b}{c}\times\frac{c}{d}\times\frac{d}{e}\times\frac{e}{f}=\frac{a}{f},$$

Or, $a^n : b^n :: a : f$.

79. When the ratio of the first of three quantities to the second is the same as the ratio of the second to the third, the ratio of the first to the second is termed the *sub-duplicate* of the ratio of the first to the third.

When four quantities are continued proportionals, the ratio of the first to the second is called the *sub-triplicate* of the ratio of the first to the fourth.

A ratio, compounded of a simple and sub-duplicate ratio, is called a *sesquiplicate ratio.*

80. The sub-duplicate ratio is equivalent to the ratio of the square roots; the sub-triplicate, to the ratio of the cube roots; and the sesquiplicate, to the ratio of the square roots of the cubes.

Let $a : b :: b : c :: c : d$. Also, $b : e :: e : c$.

First, (art. 78,) $\frac{a^2}{b^2}=\frac{a}{c}$, $\therefore \frac{a}{b}=\frac{a^{\frac{1}{2}}}{c^{\frac{1}{2}}}$ or, $a : b :: a^{\frac{1}{2}} : c^{\frac{1}{2}}$.

In like manner, $\frac{a}{b}=\frac{a^{\frac{1}{3}}}{d^{\frac{1}{3}}}$, or $a:b::a^{\frac{1}{3}}:d^{\frac{1}{3}}$.

Also, since $b:e::e:c$, and $a:b::c:d$.

$e^2=bc=ad$, (art. 69.) But $\frac{a^3}{b^3}=\frac{a}{d}=\frac{a^2}{ad}=\frac{a^2}{e^2}$,

$\therefore \frac{a^{\frac{3}{2}}}{b^{\frac{3}{2}}}=\frac{a}{e}$, or $a^{\frac{3}{2}}:b^{\frac{3}{2}}::a:e$, in which the ratio of $a:e$, is compounded of the ratios of $a:b$, and of $b:e$, or of the ratio of $a:b$, and the sub-duplicate of the ratio of $b:c$.

ON THE VARIATIONS OF QUANTITIES.

81. In the investigation of the relation which varying and dependent quantities bear to each other, the conclusions are more readily obtained, by expressing only two terms in each proportion, than by retaining the four. But though, in considering the variations of such quantities, two terms only are expressed, it must be remembered that four are supposed; and that the operations, by which our conclusions are obtained, are in reality the operations of proportionals.

82. One quantity is said to vary *directly*, as another, when one is such a function* of the other, that, if the former be changed, the latter will be changed in the same ratio. Thus, if B be such a function of A, that by changing A to a, B shall be changed to b; making $A:a::B:b$,

* The *function* of any variable quantity x, is an algebriac expression, in which x, combined with invariable quantities, is involved. Thus, $1+x$, $(1+x)^2$, ax, x^n, a^x, etc. are functions of x. Analysts sometimes use the Greek letter ϕ, to denote a function. Thus, ϕx may represent any function of x.

A is said to vary *directly* as B. This relation is designated thus, $A \propto B$.

83. One quantity is said to vary *inversely* as another, when the latter is such a function of the former, that the one being increased or diminished, the other will be diminished or increased in the same ratio. Thus, if B be such a function of A, that, by changing A to a, B becomes changed to b; making $A : a :: b : B$. Then A is said to vary inversely as B. Indicated thus, $A \propto \frac{1}{B}$.

84. One quantity is said to vary as two others jointly, when the two last are such functions of the first, that the ratio, which any two values of the first bear to each other, shall be the same as the ratio compounded of the ratios of the corresponding values of the other two. Thus, A varies as B and C jointly, ($A \propto BC$,) when A being changed to a, B changes to b, and C to c, so that $\frac{A}{a} = \frac{B}{b} \times \frac{C}{c}$.

85. One quantity is said to vary directly as a second, and inversely as a third, when the ratio which any two values of the first bear to each other, is the same as the ratio compounded of the direct ratio of the corresponding values of the second, and the inverse ratio of those of the third.

Thus, A varies directly as B, and inversely as C, ($A \propto \frac{B}{C}$,) when A, B, C; a, b, c, being corrresponding values, $\frac{A}{a} = \frac{B}{b} \times \frac{c}{C}$.

In the following articles, A, B, C, etc. represent corresponding values of any quantities, and a, b, c, etc. any other corresponding values of quantities.

86. If $A \propto B$, and m, n denote any given numbers,

Then $A \propto mB \propto \frac{B}{n}$,

For since $\frac{A}{a} = \frac{B}{b}$, and $\frac{B}{b} = \frac{mB}{mb} = \frac{\frac{1}{n}B}{\frac{1}{n}b}$; $\frac{A}{a} = \frac{mB}{mb} = \frac{\frac{1}{n}B}{\frac{1}{n}b}$.

87. If $A \propto B$, then $A^n \propto B^n$; and $A^{\frac{1}{n}} \propto B^{\frac{1}{n}}$. See art. 75.

88. If $A \propto B$, and $C \propto D$, then $AC \propto BD$. See art. 77.

89. If $A \propto BC$, then $B \propto \frac{A}{C}$, and $C \propto \frac{A}{B}$,

For $\frac{A}{a} = \frac{B}{b} \times \frac{C}{c}$. $\therefore \frac{Ac}{aC} = \frac{B}{b}$, or $\frac{B}{b} = \frac{A}{a} \times \frac{c}{C} (B \propto \frac{A}{C})$

Also, $\frac{Ab}{aB} = \frac{C}{c}$, or $\frac{C}{c} = \frac{A}{a} \times \frac{b}{B}$, $(C \propto \frac{A}{B}.)$

From the three preceding articles it appears, that quantities connected by the sign $\propto$, may be treated as the members of an equation, as far as multiplication, division, involution, or evolution is concerned.

90. If $A \propto C$, and $B \propto C$, A and B being quantities of the same kind, then $(A \pm B) \propto C$, and $\sqrt{AB} \propto C$.

For since $\frac{A}{a} = \frac{C}{c}$, and $\frac{B}{b} = \frac{C}{c}$; $\frac{A}{a} = \frac{B}{b}$. $\therefore \frac{A}{B} = \frac{a}{b}$,

And $\frac{A}{B} \pm 1 = \frac{a}{b} \pm 1$, or $\frac{A \pm B}{B} = \frac{a \pm b}{b}$. $\therefore \frac{A \pm B}{a \pm b} = \frac{B}{b} = \frac{C}{c}$

Again, since $A \propto C$, and $B \propto C$, (art. 88,) $AB \propto C^2$.

$\therefore$ (art. 87,) $\sqrt{AB} \propto C$.

91. If while A and B vary, AB=a constant quantity, $A \propto \frac{1}{B}$, and $B \propto \frac{1}{A}$

For since $AB = ab$, $\frac{A}{a} = \frac{b}{B}$, $(A \propto \frac{1}{B},)$

And $\frac{B}{b} = \frac{a}{A} (B \propto \frac{1}{A}.)$

Section IX.

OF SERIES.

92. From the nature of powers, (art. 14,) we readily discover that any two powers of the same root, multiplied together, produce a power indicated by the sum of the exponents of the factors; thus $a^2.a^3=a^5$; $a^m \times a^n = a^{m+n}$.

On the other hand, $\frac{a^5}{a^3}=\frac{1.a.a.a.a.a}{1.a.a.a}=aa=a^2$, $\frac{a^m}{a^n}=a^{m-n}$

Suppose $n=m+v$, then $\frac{a^m}{a^n}=\frac{a^m}{a^m.a^v}=\frac{1}{a^v}$

But in this case $m-n=-v$, if then $\frac{a^m}{a^n}$ can be always denoted by a^{m-n}; a^{-v} must be equivalent to $\frac{1}{a^v}$

Considering the consecutive powers of a given root as the terms of a geometrical series, extending at pleasure, above and below unity.

$$\frac{1}{a^3},\ \frac{1}{a^2},\ \frac{1}{a},\ 1,\ a,\ a^2,\ a^3, \text{ etc.}$$

Or the equivalent series, a^{-3}, a^{-2}, a^{-1}, a^0, a^1, a^2, a^3, etc.

The index denotes the order of the term beginning with unity, and making the index positive or negative, according as the quantity is above or below the point of commencement.

93. Between any two of these terms, let $n-1$ mean proportionals be interposed, ex. gr. between a^0 $(=1,)$ and a, calling the first x, then (art. 78,) $1 : a :: 1^n : x^n :: 1 : x^n$.

$$\therefore a=x^n.$$

Our series then becomes, (x^0 being $=1=a^0$.)

$$x^{-2n} \ldots x^{-n-1},\ x^{-n},\ \ldots\ x^{-2},\ x^{-1},\ x^0,\ x^1,\ x^2,$$
$$\ldots x^n,\ x^{n+1},\ x^{n+2},\ x^{2n},\ x^{2n+1}, \text{ etc.}$$

Now, the product of any two terms of this series is manifestly indicated by x, with an exponent equal to the sum of the exponents of the factors, as $x^n \times x^{2n+1} = x^{3n+1}$;

$$x^{n+1} \times x^{-n} = \frac{x^{n+1}}{1} \times \frac{1}{x^n} = \frac{x^{n+1}}{x^n} = x^1 = x^{n+1-n}.$$

$$\frac{x^{2n+3}}{x^{n+2}} = x^{n+1} = x^{2n+3-(n+2)}$$

$$\frac{x^{n+4}}{x^{-n}} = x^{n+4} \div \frac{1}{x^n} = x^{n+4} \times x^n = x^{2n+4} = x^{n+4+n}.$$

If for x and its powers we write $a^{\frac{1}{n}}$ and its powers, the series, though changed as to the form of its exponents, evidently retains the same essential character, and the operations of multiplication and division are performed by taking the sum and difference of the exponents of the factors, whether those exponents are positive or negative, integral or fractional. By thus arranging the powers as the terms of a geometrical series, we perceive that the exponent, whether integral or fractional, serves not only to designate the power, but to indicate the situation of the term, in relation to the unit's place.

94. From these principles we readily infer, that any letter or quantity may be removed from the denominator of a fraction to its numerator, and vice versa, by changing the sign of its exponent.

Thus, $\frac{a}{x^2} = \frac{a}{1} \times \frac{1}{x^2} = ax^{-2}$.

$$\frac{a+b}{(a^2-b^2)^{\frac{1}{2}}} = (a+b).(a^2-b^2)^{-\frac{1}{2}}. \qquad ab^3 = \frac{a}{b^{-3}}$$

95. It frequently happens in the division or evolution of algebriac quantities, as well as in common arithmetic, that to whatever extent the process may be continued, a remainder will still occur; in which case the resulting quotient or root mostly assumes the form of an infinite series.

Thus, $\frac{1}{1-x}=1+x+x^2+x^3+x^4+$, etc. to infinity.

$$\sqrt{a^2+b^2}=a+\frac{b^2}{2a}-\frac{b^4}{8a^3}+\frac{b^6}{16a^5}-\frac{5b^8}{128a^7}+, \text{ etc.}^*$$

In some instances, when a few terms of the series are obtained, *the law of continuation*, or the relation of the successive terms to those which precede them, becomes manifest. The first of the series above given may be readily continued to any proposed extent. The law of continuation in the last is not obvious on first view; it will however, be shown further on.

96. In the investigations connected with series, the method of *indeterminate co-efficients* is often found particularly convenient. It depends upon the following theorem.

Let $Ax+Bx^2+Cx^3+Dx^4$, etc. $=ax+bx^2+cx^3+dx^4$, etc., the series being both infinite, or, if finite, extending to the same number of terms; and A, B, C, a, b, c, invariable; if then the above equation be true, whatever value may be assigned to x, A will be equal to a, $B=b$, etc.; for dividing by x, we have

$$A+Bx+Cx^2+Dx^3, \text{ etc.}=a+bx+cx^2+dx^3, \text{ etc.}$$

Now, the equation being true for all values of x, must hold if $x=0$, in which case the equation becomes $A=a$; subtracting this equation from the given one,

$$Bx+Cx^2+Dx^3, \text{ etc.}=bx+cx^2+dx^3, \text{ etc.}$$

Dividing by x, $B+Cx+Dx^2, \text{ etc.}=b+cx+dx^2, \text{ etc.}$

Whence, if $x=0$, $B=b$.

In the same manner $C=c$, $D=d$, etc.

This equation becomes by transposition,

$$\left.\begin{matrix} Ax+Bx^2+Cx^3+Dx^4+, \text{ etc.} \\ -ax-bx^2-cx^3-dx^4+, \text{ etc.} \end{matrix}\right\}=0.$$

* When the successive terms of an infinite series continually decrease, it is called a converging series; in which case, the sum of the series may be approximated by collecting a finite number of terms.

Hence it appears, that when all the terms of a general equation are brought to one side, the co-efficients of the several powers of the unknown quantity are respectively equal to 0.

This principle is applied in the following examples.

1. Required to express $\frac{1}{1-2x+x^2}$ in a series.

It is easy to perceive that the first term must be 1.

Assume then $\frac{1}{1-2x+x^2}=1+ax+bx^2+cx^3+dx^4$, etc.

Multiply by $1-2x+x^2$, and bring the terms to one side of the equation

$$\left.\begin{array}{rrrrrl} \text{Then, } 1+ax+ & bx^2 & +cx^3 & +dx^4 & +, & \text{etc.} \\ -2x- & 2ax^2- & 2bx^3- & 2cx^4- & , & \text{etc.} \\ -1 & +x^2 & +ax^3 & +bx^4 & +, & \text{etc.} \end{array}\right\}=0$$

Hence, $a-2=0$, $b-2a+1=0$, $c-2b+a=0$, $d-2c+b=0$.

$\therefore a=2$, $b=4-1=3$, $c=6-2=4$, $d=8-3=5$.

Whence, $\frac{1}{1-2x+x^2}=1+2x+3x^2+4x^3+5x^4+$, etc. in which the *law of continuation* is manifest.

2. Required to develope $\sqrt{a^2+x^2}$, in a series.

$$\sqrt{a^2+x^2}=\sqrt{(a^2\times\overline{1+\frac{x^2}{a^2}})}=a\sqrt{(1+\frac{x^2}{a^2})}=a\sqrt{1+z^2},$$

Putting $\frac{x^2}{a^2}=z^2$.

Assume $\sqrt{1+z^2}=1+az^2+bz^4+cz^6+dz^8+$, etc.*

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* If the series $1+az+bz^2+cz^3$, etc. had been assumed, we should have found $a=0$, $c=0$, etc.

By involution and transposition,

$$\left.\begin{array}{r} 1+2az^2+2bz^4+2cz^6+2dz^8+2ez^{10}+, \text{ etc.} \\ a^2z^4+2abz^6+2acz^8+2adz^{10}, \text{ etc.} \\ b^2z^8+2bcz^{10}, \text{ etc.} \end{array}\right\} = 0.$$

$-1-z^n$.

$\therefore 2a-1=0,\ 2b+a^2=0,\ 2c+2ab=0,\ 2d+2ac+b^2=0$, etc.

Whence, $a=\frac{1}{2}, b=-\frac{1}{8}, c=\frac{1}{16}, d=-\frac{5}{128}, e=\frac{7}{256}$

$$\therefore \sqrt{1+z^2}=1+\frac{1}{2}z^2-\frac{1}{8}z^4+\frac{1}{16}z^6-\frac{5}{128}z^8+\frac{7}{256}z^{10}, \text{ etc.}$$

Consequently,

$$\sqrt{a^2+x^2}=a\left\{1+\frac{x^2}{2a^2}-\frac{x^4}{8a^4}+\frac{x^6}{16a^6}-\frac{5x^8}{128a^8}, \text{ etc.}\right\}=$$

$$a+\frac{x^2}{2a}-\frac{x^4}{8a^3}+\frac{x^6}{16a^5}-\frac{5x^8}{128a^7}+\frac{7x^{10}}{256a^9}, \text{ etc.}$$

To find, if possible, the law of continuation, we observe that

$$-\frac{1}{8}=\frac{1}{2}\times\frac{-1}{4},\quad \frac{1}{16}=\frac{1}{2}\times\frac{-1}{4}\times\frac{-3}{6},$$

$$-\frac{5}{128}=\frac{1}{2}\times\frac{-1}{4}\times\frac{-3}{6}\times\frac{-5}{8},\quad \frac{7}{256}=\frac{1}{2}\times\frac{-1}{4}\times\frac{-3}{6}\times\frac{-5}{8}\times\frac{-7}{10}.$$

Hence the law is evident

97. Required to express $\frac{x^n-y^n}{x-y}$ in a series, n being an integral number.

$$\frac{x^n-y^n}{x-y}=\frac{x^n}{x}\times\frac{1-\frac{y^n}{x^n}}{1-\frac{y}{x}}=x^{n-1}\times\frac{1-z^n}{1-z}$$

Assuming $z=\frac{y}{x}$.

Now, if $\frac{1-z^n}{1-z}$ be resolved into a series, it is manifest the first term will be 1.

And therefore, $\frac{x^n-y^n}{x-y}=x^{n-1}\times\frac{1-z^n}{1-z}$ resolved into a series, will have its first term x^{n-1}.

Again, $\frac{x^n-y^n}{x-y}=\frac{y^n-x^n}{y-x}=\frac{y^n}{y}\times\frac{1-\frac{x^n}{y^n}}{1-\frac{x}{y}}=y^{n-1}\times\frac{1-v^n}{1-v}$,

making $v=\frac{x}{y}$, which, resolved into a series, will have its first term y^{n-1}.

Whence it appears, that if $\frac{x^n-y^n}{x-y}$ be resolved into a series, the first and last terms will be x^{n-1}, and y^{n-1} respectively.

Assume then $\frac{1-z^n}{1-z}=1+az+bz^2+....pz^{n-2}+qz^{n-1}$.

Multiply by $(1-z,)$ and transpose $1-z^n$, whence,

$$\left.\begin{array}{ll} 1+az+bz^2+cz^3, & pz^{n-2}+qz^{n-1}, \\ -z-az^2-bz^3, & -mz^{n-2}-pz^{n-1}-qz^n, \\ -1 & +z^n, \end{array}\right\}=0$$

$\therefore$ (art. 96,) $a-1=0$, $b-a=0$, $c-b=0$, $p-m=0$, $q-p=0$, $1-q=0$. Or $a=1$, $b=1$, $c=1$, etc.

$$\therefore \frac{x^n-y^n}{x-y}=x^{n-1}\times\{1+z+z^2+z^3.\ \ z^{n-2}+z^{n-1}\}=$$

$$x^{n-1}+x^{n-2}y+x^{n-3}y^2+x^{n-4}y^3.\ \ x^{n-n-1}y^{n-2}+y^{n-1}.$$

Whence $x^n - y^n = (x-y) \times \{x^{n-1} + x^{n-2}y + x^{n-3}y^2 + y^{n-1},\}$

Here the indices of x descending regularly from $n-1$, to 0, or $n-n$, it appears the number of terms in the series is n.

98. Required to develope $(1+x)^n$ in a series, n being a whole positive number.

It is easy to perceive that the first term of the series is 1; and that the powers of x regularly ascend.

Assume then,

$(1+x)^n = 1 + ax + bx^2 + cx^3 + dx^4 +$, etc. $(A.)$

Consequently, $(1+y)^n = 1 + ay + by^2 + cy^3 + dy^4$, etc.

By subtraction, $(1+x)^n - (1+y)^n = a(x-y) + b(x^2-y^2) + c(x^3-y^3) + d(x^4-y^4,)$ etc.

Divide by $\overline{1+x} - \overline{1+y} = x-y$, and we shall have

$$\frac{(1+x)^n - (1+y)^n}{x-y} = a + b\left(\frac{x^2-y^2}{x-y}\right) + c\left(\frac{x^3-y^3}{x-y}\right) + d\left(\frac{x^4-y^4}{x-y}\right) \text{ etc.}$$

Or $(1+x)^{n-1} + (1+x)^{n-2} \times (1+y) \ldots\ldots (1+y)^{n-1} = a + b(x+y) + c(x^2+xy+y^2) + d(x^3+x^2y+xy^3+y^3) +$, etc.

This equation being true, whatever value may be assigned to x and y, let us suppose $x=y$, then this equation becomes

(art. 97,) $n.(1+x)^{n-1} = a + 2bx + 3cx^2 + 4dx^3 + 5ex^4$, etc.

Multiply by $(1+x)$

$\therefore n.(1+x)^n = a + 2bx + 3cx^2 + 4dx^3 + 5ex^4$, etc.
$\quad ax + 2bx^2 + 3cx^3 + 4dx^4$, etc.

But (eq. A,) multiplied by n,

$n.(1+x)^n = n + nax + nbx^2 + ncx^3 + ndx^4$, etc.

Consequently, $a=n$, $b = a.\frac{n-1}{2} = n.\frac{n-1}{2}$,

$$c=b.\frac{n-2}{3}=n.\frac{n-1}{2}\frac{n-2}{3};\quad d=c\frac{n-3}{4}=n.\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4};$$

$$e=d.\frac{n-4}{5}=n.\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}\cdot\frac{n-4}{5}.$$

Hence, $(a+b)^n=a^n\times(1+\frac{b}{a})^n=$

$$a^n.\{1+n\frac{b}{a}+n.\frac{n-1}{2}\frac{b^2}{a^2}+,\text{ etc.}\}=$$

$$a^n+na^{n-1}b+n.\frac{n-1}{2}a^{n-2}b^2+n.\frac{n-1}{2}\cdot\frac{n-2}{3}a^{n-3}b^3+,\text{ etc.}$$

When n is a whole positive number, as here supposed, the series is finite; the number of terms being $n+1$; for the co-efficient of term $\overline{n+2}$ is

$$n.\frac{n-1}{2}\cdot\frac{n-2}{3}\cdots\frac{n-n}{n+1}=0.$$

If $(a-b)^n$ be required, we may put $-b$ and its powers for $+b$ and its powers.

Whence, $(a-b)^n=a^n-na^{n-1}b+n\frac{n-1}{2}a^{n-2}b^2$

$$-n.\frac{n-1}{2}\cdot\frac{n-2}{3}a^{n-3}b^3+n.\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}a^{n-4}b^4,\text{ etc.}$$

99. Required to develope $(1+x)^{\frac{m}{n}}$ in a series.

Assume $(1+x)^{\frac{m}{n}}=1+ax+bx^2+cx^3+dx^4+ex^5$, etc. $(A.)$

Then $(1+y)^{\frac{m}{n}}=1+ay+by^2+cy^3+dy+ey^5$, etc. $(B.)$

Put $(1+x)^{\frac{1}{n}}=v$, $(1+y)^{\frac{1}{n}}=w$, then $1+x=v^n$, $1+y=w^n$,

$(1+x)^{\frac{m}{n}}=v^m$, $(1+y)^{\frac{m}{n}}=w^m$.

Substituting these expressions in equations A and B, and subtracting, we have

$$v^m-w^m=a(x-y)+b(x^2-y^2)+c(x^3-y^3)+$$
$$d.(x^4-y^4)+,\text{ etc.}$$

$$\therefore \text{(art. 97,) } (v-w)\times\{v^{m-1}+v^{m-2}w.\quad w^{m-1}\}=$$
$$(x-y)\,.\,\{a+b(x+y)+c.(x^2+xy+y^2)+$$
$$d(x^3+x^2y+xy^2+y^3),\text{ etc.}\}\ (C.)$$

But $x-y=\overline{1+x}-\overline{1+y}=v^n-w^n=(v-w)\times$
$$\{v^{n-1}+v^{n-2}w+.\quad w^{n-1}\}$$

$\therefore$ Dividing equation C by this, we have

$$\frac{v^{m-1}+v^{m-2}w+v^{m-3}w^2.\quad w^{m-1}}{v^{n-1}+v^{n-2}w+v^{n-3}w^2.\quad w^{n-1}}=a+b(x+y)\times$$
$$c(x^2+xy+y^2)+d(x^3+x^2y+xy^2+y^3),\text{ etc.}$$

Now, suppose $x=y$, whence $v=w$ and our equation becomes (art. 97,)

$$\frac{mv^{m-1}}{nv^{n-1}}=(\frac{m}{n}\cdot\frac{v^m}{v^n})=a+2bx+3cx^2+4dx^3+5ex^4,\text{ etc.}$$

Multiply by $v^n=1+x$, and we have

$$\frac{m}{n}v^m=\frac{m}{n}.(1+x)^{\frac{m}{n}}=a+2bx+3cx^2+4dx^3+5ex^4,\text{ etc.}$$
$$ax+2bx^2+3cx^3+4dx^4,\text{ etc.}$$

But (equation A,)

$$\frac{m}{n}.(1+x)^{\frac{m}{n}}=\frac{m}{n}+\frac{m}{n}ax+\frac{m}{n}bx^2+\frac{m}{n}cx^3+\frac{m}{n}dx^4,\text{ etc.}$$

Hence, by equating the homologous co-efficients,

$$a=\frac{m}{n};b=a.\frac{\frac{m}{n}-1}{2}=\frac{m}{n}\cdot\frac{m-n}{2n};\ c=b.\frac{\frac{m}{n}-2}{3}=$$

$$\frac{m}{n}\cdot\frac{m-n}{2n}\cdot\frac{m-2n}{3n};\ d=c.\frac{\frac{m}{n}-3}{4}=\frac{m}{n}\cdot\frac{m-n}{2n}\cdot\frac{m-2n}{3n}\cdot\frac{m-3n}{4n}$$

Hence, $(a+b)^{\frac{m}{n}} = a^{\frac{m}{n}}.(1+\frac{b}{a})^{\frac{m}{n}} =$

$$a^{\frac{m}{n}}\{1+\frac{m}{n}.\frac{b}{a}+\frac{m}{n}.\frac{m-n}{2n}.\frac{b^2}{a^2}+\frac{m}{n}.\frac{m-n}{2n}.\frac{m-2n}{3n}.\frac{b^3}{a^3}, \text{ etc.}\}$$

100. In the preceding investigation m has been supposed to be affirmative; if we now take m negative, and make the same assumptions as before, we have

$$v^{-m}-w^{-m} = \frac{1}{v^m}-\frac{1}{w^m} = \frac{w^m-v^m}{v^m.w^m} = -\frac{1}{v^m.w^m}.\overline{v^m-w^m} =$$

$$-\frac{1}{v^m w^m}.(v-w).\{v^{m-1}+v^{m-2}w+v^{m-3}w^2. \quad w^{m-1}\} =$$

$$(x-y).\{a+b(x+y)+c.(x^2+xy+y^2)+$$
$$d(x^3+x^2y+xy^2+y^3)+, \text{ etc.}\} =$$
$$(v-w).\{v^{n-1}+v^{n-2}w. \quad w^{n-1}\}$$
$$\{a+b(x+y)+c(x^2+xy+y^2)+, \text{ etc.}\}$$

Whence, dividing by $n.\overline{v-w}$, and assuming $x=y$,

$$-\frac{1}{.v^{2m}}\times\frac{mv^{m-1}}{n} = v^{n-1}\{a+2bx+3cx^2+4dx^3+, \text{ etc.}\}$$

$\therefore$ Multiplying by v, and putting $1+x$ for v^n,

$$-\frac{m}{n}v^{-m} = \overline{1+x}\{a+2bx+3cx^2+4dx^3+, \text{ etc.}\} =$$
$$a+2bx+3cx^2+4dx^3, \text{ etc.}$$
$$ax+2bx^2+3cx^3, \text{ etc.}$$

Now, multiplying equation A by $\frac{-m}{n}$, and equating the homologous co-efficients,

$$a = \frac{-m}{n}, \quad b = \frac{-m}{n}\times\frac{-m-n}{2n},$$
$$c = \frac{-m}{n}\times\frac{-m-n}{2n}\times\frac{-m-2n}{3n}, \text{ etc.}$$

Whence it appears that the formula is correct whether the index is positive or negative. This is Newton's celebrated binomial theorem.

101. By the aid of this theorem, a binomial may be raised to any power, or evolved to any root by simple substitution.

Several examples of the use of this theorem, when the index is a whole positive number, are given in section second.

1. Required to develope $\sqrt[3]{\frac{a^5}{(a+x)^2}}$ in a series.

$$\sqrt[3]{\frac{a^5}{(a+x)^2}} = \sqrt[3]{\frac{a^3}{(1+\frac{x}{a})^2}} = a\left(1+\frac{x}{a}\right)^{-\frac{2}{3}}$$

Here $m=-2$, $n=3$.

$$\therefore a\left(1+\frac{x}{a}\right)^{-\frac{2}{3}} = a\left\{1-\frac{2}{3}\cdot\frac{x}{a}+\frac{2}{3}\cdot\frac{-2-3}{6}\cdot\frac{x^2}{a^2}+\frac{2}{3}\cdot\frac{-2-3}{6}\cdot\right.$$

$$\left.\frac{-2-6}{9}\cdot\frac{x^3}{a^3}+\frac{2}{3}\cdot\frac{-2-3}{6}\cdot\frac{-2-6}{9}\cdot\frac{-2-9}{12}\cdot\frac{x^4}{a^4}, \text{ etc.}\right\} =$$

$$a-\frac{2x}{3}+\frac{5x^2}{9a}-\frac{40x^3}{81a^2}, +\frac{110x^4}{243a^3}, \text{ etc.}$$

2. What is the value of $\sqrt[3]{a^3+b}$ in a series?

$$\text{Ans. } a+\frac{b}{3a^2}-\frac{b^2}{9a^5}+\frac{5b^3}{81a^8}-\frac{10b^4}{243a^{11}}, \text{ etc}$$

3. What is the value of $\sqrt{\frac{1}{a^2-x^2}}$ in a series.

$$\text{Ans. } \frac{1}{a}+\frac{x^2}{2a^3}+\frac{3x^4}{8a^5}+\frac{5x^6}{16a^7}+\frac{35x^8}{128a^9}, \text{ etc.}$$

4. Required the value of $(a-b)^{\frac{2}{5}}$ in a series.

Result, $a^{\frac{2}{5}} \cdot \{ 1-\frac{2b}{5a}-\frac{3b^2}{25a^2}-\frac{8b^3}{125a^3}$

$-\frac{26b^4}{625a^4}-\frac{468b^5}{15625a^5}$, etc.$\}$

5. Required $\sqrt{\frac{a+x}{a-x}}$ in a series.*

Result, $1+\frac{x}{a}+\frac{x^2}{2a^2}+\frac{x^3}{2a^3}+\frac{3x^4}{8a^4}+\frac{3x^5}{8a^5}$, etc.

6. What is $(\frac{a}{b-x})^{\frac{3}{5}}$ in a series?

Ans. $(\frac{a}{b})^{\frac{3}{5}} \cdot \{ 1+\frac{3x}{5b}+\frac{12x^2}{25b^2}+\frac{52x^3}{125b^3}+\frac{234x^4}{625b^4}$ etc,$\}$

REVERSION OF SERIES.

102. Given $z=x+\frac{x^2}{2}+\frac{x^3}{2.3}+\frac{x^4}{2.3.4}+\frac{x^5}{2.3.4.5}$, etc. to infinity, to find x in terms of z.

---

* This quantity is reducible to

$$\sqrt{\frac{(a+x)^2}{a^2-x^2}}=\sqrt{\frac{(1+\frac{x}{a})^2}{1-\frac{x^2}{a^2}}}=(1+\frac{x}{a}).(1-\frac{x^2}{a^2})^{-\frac{1}{2}}$$

$$\left.\begin{array}{rl} \text{Assume } x=az+bz^2+cz^3+dz^4+, \text{ etc.} & \\ \text{Then } \frac{x^2}{2}= & \frac{a^2z^2}{2}+abz^3+acz^4, \text{ etc.} \\ & \frac{b^2z^4}{2}, \text{ etc.} \\ \frac{x^3}{2.3}= & \frac{a^3z^3}{2.3}+\frac{a^2b}{2}z^4, \text{ etc.} \\ \frac{x^4}{2.3.4}= & \frac{a^4}{2.3.4}z^4, \\ -z. & \end{array}\right\}=0$$

$\because a=1, b=-\frac{1}{2}, c=\frac{1}{2}-\frac{1}{6}=\frac{1}{3}, d=-\frac{1}{3}-\frac{1}{8}+\frac{1}{4}-\frac{1}{24}=-\frac{1}{4}.$

Whence, $x=z-\frac{z^2}{2}+\frac{z^3}{3}-\frac{z^4}{4}+$, etc. where the law of continuation is manifest.

2. Given $x=z-\frac{z^3}{2.3}+\frac{z^5}{2.3.4.5}-\frac{z^7}{2.3.4.5.6.7}$, etc. to find z in terms of x.

$$\left.\begin{array}{rl} \text{Let } z=ax+bx^3+cx^5+dx^7 & \\ \text{Then } -\frac{z^3}{2.3}= & -\frac{a^3}{2.3}x^3-\frac{a^2b}{2}.x^5-\frac{ab^2}{2}x^7. \\ & -\frac{a^2c}{2}x^7. \\ \frac{z^5}{2.3.4.5}= & \frac{a^5}{2.3.4.5}x^5+\frac{a^4b}{2.3.4}x^7. \\ -\frac{z^7}{2.3.4.5.6.7}= & -\frac{a^7}{2.3.4.5.6.7}x^7. \\ -x. & \end{array}\right\}=0$$

$$\therefore a=1, b=\frac{1}{2.3}, c=\frac{1}{2.3.2}-\frac{1}{2.3.4.5}=\frac{1.3}{2.4.5}$$

$$d=\frac{1}{2.4.9}+\frac{1.3}{2.8.5}-\frac{1}{4.9.4}+\frac{1}{2.3.4.5.6.7}=\frac{1.3.5}{2.4.6.7}$$

Hence, $z=x+\frac{x^3}{2.3}+\frac{1.3x^5}{2.4.5}+\frac{1.3.5x^7}{2.4.6.7}+\frac{1.3.5.7x^9}{2.4.6.8.9}$, etc.

3. Required to revert the general series $x=ay+by^2+cy^3+dy^4$, etc.

Assume $y=Ax+Bx^2+Cx^3+Dx^4$, etc.

$$\left.\begin{array}{rl} ay= & Aax+Bax^2+Cax^3+Dax^4, \text{ etc.} \\ by^2= & bA^2x^2+2bABx^3+(B^2b+2ACb)x^4, \text{ etc.} \\ cy^3= & cA^3x^3+3A^2Bcx^4, \text{ etc.} \\ dy^4= & dA^4x^4, \text{ etc.} \\ -x. & \end{array}\right\}=0$$

$$\therefore \text{(art. 96,)}\ A=\frac{1}{a},\quad B=-\frac{b}{a^3},\quad C=\frac{2b^2-ac}{a^5}$$

$$D=-\frac{5b^3-5abc+a^2d}{a^7}$$

And $y=\frac{x}{a}-\frac{bx^2}{a^3}+\frac{2b^2-ac}{a^5}x^3-\frac{5b^3-5abc+a^2d}{a^7}x^4$, etc.

4. Given $z=u-\frac{u^3}{3}+\frac{u^5}{5}-\frac{u^7}{7}+\frac{u^9}{9}+$, etc. to infinity, to find u in terms of z.

Result, $u=z+\frac{1}{3}z^3+\frac{2}{15}z^5+\frac{17}{315}z^7$, etc.

5. Given $x=y-\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^4}{4}+\frac{y^5}{5}-$, etc. to infinity, to find y in terms of x.

Result, $y=x+\frac{x^2}{2}+\frac{x^3}{2.3}+\frac{x^4}{2.3.4}+\frac{x^5}{2.3.4.5}$, etc.

SUMMATION OF SERIES.

103. Infinite series are sometimes of such nature, that a quantity can be found, to which the series continually approximates, and to which, without attaining perfect equality, it arrives more nearly than by any assignable difference. The quantity to which the series, by continued extension, thus approximates, is called the sum of the infinite series. Thus we say, .3333, etc. to infinity, $=\frac{1}{3}$; for no number less than $\frac{1}{3}$ can be assigned, which the series may not, by extension, be made to exceed.

Required the sum of $1+x+x^2+x^3$, etc. to infinity, x being supposed <1.*

Put $y=1+x+x^2+x^3$, etc. to infinity.

$\therefore xy=x+x^2+x^3$, etc. to infinity.

And $y-xy=1$. $\therefore y=\frac{1}{1-x}$.

2. Required the sum of $1+2x+3x^2+4x^3+5x^4+$, etc. to infinity, supposing $x<1$.

Put $y=1+2x+3x^2+4x^3+5x^4$, etc.

$\therefore -2xy=-2x-4x^2-6x^3-8x^4$, etc.

And $x^2y=x^2+2x^3+3x^4$, etc.

Adding these three equations together

$$y-2xy+x^2y=1.$$

$$\therefore y=\frac{1}{1-2x+x^2}=\frac{1}{(1-x)^2}.$$

3. Required the sum of $\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}$, etc. to infinity.

---

*The character $<$ is used to express inequality, the opening being presented to the greater quantity.

Assume $x=1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+$, etc. to infinity.

Then $x-1=\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}$, etc.

Subtract the latter from the former, and

$1=\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}$, etc. the sum required.

4. Required the sum of $\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}$, to infinity.

By last example, $1=\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}$, etc.

Subtract $\frac{1}{1.2}$. $\therefore \frac{1}{2}=\frac{1}{2.3}+\frac{1}{3.4}+\frac{1}{4.5}+\frac{1}{5.6}$, etc.

By subtraction, $\frac{1}{2}=\frac{2}{1.2.3}+\frac{2}{2.3.4}+\frac{2}{3.4.5}+\frac{2}{4.5.6}$, etc.

Whence, $\frac{1}{4}=\frac{1}{1.2.3}+\frac{1}{2.3.4}+\frac{1}{3.4.5}+\frac{1}{4.5.6}$, etc. the sum required.

The student who desires to pursue this subject, may consult Wood's Algebra, Article 411, etc., or Young's Algebra, page 251 etc.

DIFFERENTIAL METHOD.

104. In any series of quantities, a, b, c, d, e, etc., if each term be subtracted from the next following one, and each term of the series of differences be taken from the next, and so on, the following series will be obtained.

1st differences, $b-a$, $c-b$, $d-c$, $e-d$, etc.

2d diff. $c-2b+a$, $d-2c+b$, $e-2d+c$, etc.

3d diff. $d-3c+3b-a$, $e-3d+3c-b$, etc.
4th diff. $e-4d+6c-4b+a$, etc.
Or these, 1st diff. $-a+b$, $-b+c$, $-c+d$, etc.
2d diff. $a-2b+c$, $b-2c+d$, etc.
3d diff. $-a+3b-3c+d$, $-b+3c-3d+e$, etc.
4th diff. $a-4b+6c-4d+e$,
5th diff. $-a+5b-10c+10d-5e+f$, etc.

By a little attention to these expressions, and a comparison of their formation with the powers of a binomial, we perceive that if A be taken to denote the first term of any (the nth) order of differences.

$$\pm A = a - nb + n.\frac{n-1}{2}c - n.\frac{n-1}{2}.\frac{n-2}{3}d, \text{ etc.}$$

the sign + being used when n is an even number, and − when n is odd.

If the differences of any order vanish, any one of the terms may be found by means of the others.

Suppose the 4th difference $a-4b+6c-4d+e=0$, and c was not known, we should find,

$$c = \frac{4(b+d)-(a+e)}{6}.*$$

105 Let D_1, D_2, D_3, D_4, etc. denote the first terms of the 1st, 2d, 3d, 4th, etc. orders of differences, viz: $D_1 = -a+b$, $D_2 = a-2b+c$, $D_3 = -a+3b-3c+d$, $D_4 = a-4b+6c-4d+e$, etc.; then we find $b = a+D_1$, $c = a+2D_1+D_2$, $d = a+3D_1+3D_2+D_3$, $e = a+4D_1+6D_2+4D_3+D_4$.

* By this method the computers of the Nautical Almanac verify their calculations of the moon's longitude, latitude, etc.

Hence, the $n+1$th term of the series

$$=a+nD_1+n.\frac{n-1}{2}D_2+n.\frac{n-1}{2}.\frac{n-2}{3}D_3, \text{ etc.}$$

Consequently the nth term

$$=a+\overline{n-1}.D_1+\overline{n-1}.\frac{n-2}{2}D_2+\overline{n-1}.\frac{n-2}{2}.\frac{n-3}{3}D_3, \text{ etc.}$$

1. Required the nth term of the series of odd numbers, 1, 3, 5, 7, etc.

Here the 1st diff. are 2, 2, etc.; 2d diff. 0.

$\therefore$ the nth term $=1+2.\overline{n-1}=2n-1$.

106. Required the sum of n terms of the series a, b, c, d, etc.

This is manifestly the same as the $\overline{n+1}$th term of the following:

$$0,\ 0+a,\ 0+a+b,\ 0+a+b+c, \text{ etc.}$$

Hence, by the last article, the required sum

$$=na+n.\frac{n-1}{2}D_1+n.\frac{n-1}{2}.\frac{n-2}{3}D_2+n.\frac{n-1}{2}.\frac{n-2}{3}.\frac{n-3}{4}D_3,$$

etc.

1. Required the sum of n terms of the square numbers, 1, 4, 9, 16, etc.

Here $a=1$, $D_1=3$, $D_2=2$, $D_3=0$.

$\therefore$ the sum

$$=n+n.\frac{n-1}{2}.3+n.\frac{n-1}{2}.\frac{n-2}{3}.2=\frac{n.\overline{n+1}.\overline{2n+1}}{1.2.3.}$$

2. Required the sum of the cube numbers, 1, 8, 27, 64, etc. continued to n terms.

Ans. $\left(n.\frac{n+1}{2}\right)^2$.

3. Required the sum of 25 terms of the series, 1, 3, 5, 7, etc. Ans. 625.

4. Required the sum of 15 terms of the series 1, 16, 81, 256, 625. 1296, atc. Ans. 178312.

When the differences at length vanish, any term of the series, or the sum of any number of terms, may be *accurately determined* by the methods used in this and the preceding articles: when the differences become small, but do not vanish, a near approximation can be made.

107. In article 105, the number n is supposed to be an integer, in which case, if the differences vanish, the rule is demonstrably correct; the same formula is, however, applied to the case where n is fractional.

Suppose a series p, q, r, s, etc., of equi-different quantities, and another series, a, b, c, d, etc., such that the terms of the latter shall be similar functions of the correlative terms of the former. It is proposed to find a term y, in the latter series, corresponding to v in the former; $v-p$ being given.

Let $q-p : v-p :: 1 : x$, and take as before, $\underset{1}{D}$, $\underset{2}{D}$, $\underset{3}{D}$, etc. the first of the 1st, 2d, 3d, etc. differences.

Then assuming the series, (art. 105.)

$$y=a+x\underset{1}{D}+x.\frac{x-1}{2}\underset{2}{D}+x.\frac{x-1}{2}.\frac{x-2}{3}\underset{3}{D}, \text{ etc.}$$

This gives, when $x=0$, $y=a$, when $x=1$, $y=a+\underset{1}{D}$, when $x=2$, $y=a+2\underset{1}{D}+\underset{2}{D}$. etc.; but these expressions equal b, c, etc. as they ought to do. This formula, being demonstrably correct when x is integral, is assumed as a convenient and useful approximation when x is a fraction.

Required to find the $6\frac{1}{2}$th term in the series 1, 4, 9, 16.

Here $x=\frac{11}{2}$ $a=1$, $\underset{1}{D}=3$, $\underset{2}{D}=2$, $\underset{3}{D}=0$.

$$\therefore y=1+\frac{11}{2}\times\frac{3}{1}+\frac{11}{2}\times\frac{9}{4}\times\frac{2}{1}=\frac{169}{4}$$

2. Let a, b, c, d, be arcs of a great circle intercepted between a fixed star and the moon's centre at noon and midnight of two successive days, it is required to find the distance at 15 hours from the first noon.

Here $12:15::1:x=\frac{5}{4}$

$$\therefore \text{dist. } (y)=a+\frac{5}{4}D_1+\frac{5}{4}\times\frac{1}{8}D_2+\frac{5}{4}\times\frac{1}{8}\times\frac{-3}{12}D_3=$$

$$a+\frac{5}{4}D_1+\frac{5}{32}D_2-\frac{5}{128}D_3.$$

CONSTRUCTION OF LOGARITHMS.

108. From article 92, it is obvious that, a being any number, if $a^n=\mathcal{N}$, and $a^m=M$, $a^{n+m}=\mathcal{N}M$, and $a^{n-m}=\frac{\mathcal{N}}{M}$. If, therefore, all the numbers used in calculation were expressed in powers of a, multiplication and division might be performed by the addition and subtraction of the exponents of a. These exponents, thus employed, are termed *logarithms*. When $a^n=\mathcal{N}$, n is the logarithm of $\mathcal{N}$.

Let x be such number, that a^x may denote any proposed number.

Put $a=1+b$, then (art. 98, 99, 100,)

$$(1+b)^x=1+xb+x.\frac{x-1}{2}b^2+x.\frac{x-1}{2}.\frac{x-2}{3}b^3+x.\frac{x-1}{2}\cdot\frac{x-2}{3}\cdot\frac{x-3}{4}b^4, \text{ etc.}$$

$$=1+bx+b^2\left\{\frac{x^2}{2}-\frac{x}{2}\right\}+$$

$$b^3\left\{\frac{x^3}{6}-\frac{x^2}{2}+\frac{x}{3}\right\}+b^4\left\{\frac{x^4}{24}-\frac{x^3}{4}+\frac{11x^2}{24}-\frac{x}{4}\right\}\text{ etc.}$$

If, therefore, we collect the terms containing the successive powers of x, and denote the above equation thus,

$$(1+b)^x = 1 + Ax + Bx^2 + Cx^3 + Dx^4, \text{ etc.}$$

We readily find that

$$A = b - \frac{b^2}{2} + \frac{b^3}{3} - \frac{b^4}{4} + \frac{b^5}{5}, \text{ etc. } (Q.)$$

But the values of B, C, etc. remain to be determined.

If for x in the above equation we substitute $2x$, we shall have

$$(1+b)^{2x} = 1 + 2Ax + 4Bx^2 + 8Cx^3 + 16Dx^4, \text{ etc.}$$

Squaring the same equation,

$$\begin{aligned}(1+b)^{2x} = 1 &+ 2Ax + 2Bx^2 + 2Cx^3 + 2Dx^4, \text{ etc.} \\ &A^2x^2 + 2ABx^3 + 2ACx^4, \text{ etc.} \\ &\qquad B^2x^4, \text{ etc.}\end{aligned}$$

Comparing the co-efficients of like powers of x.

$$B = \frac{A^2}{2},\ C = \frac{A^3}{2.3}.D = \frac{A^4}{2.3.4}, \text{ etc}$$

$$\therefore a^x = (1+b)^x = 1 + Ax + \frac{A^2x^2}{2} + \frac{A^3x^3}{2.3} + \frac{A^4x^4}{2.3.4}, \text{ etc. } (R.)$$

In like manner,

$$\frac{1}{(1-b)^x} = (1-b)^{-x} = 1 + xb + x.\frac{x+1}{2}b^2 + x.\frac{x+1}{2}.\frac{x+2}{3}b^3, \text{ etc.}$$

$$= 1 + Ax + \frac{A^2x^2}{2} + \frac{A^3x^3}{2.3} + \frac{A^4x^4}{2.3.4}, \text{ etc. } (S.) \text{ in which}$$

$$A = b + \frac{b^2}{2} + \frac{b^3}{3} + \frac{b^4}{4}, \text{ etc. } (I)$$

109. Let $1+y$ denote a number whose logarithm is required; then $1+y=(1+b)^x$.

$\therefore (1+y)^z=(1+b)^{xz}$, z being any number whatever.

Here x is the logarithm of $(1+b)^x$, or of a^x, or $1+y$.

But $(1+y)^z=1+Vz+\frac{V^2z^2}{2}+\frac{V^3z^3}{2.3}$, etc. (*R.* 108.)

V being $=y-\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^4}{4}+\frac{y^5}{5}$, etc. (*Q.* 108.)

Also $(1+b)^{xz}=1+Axz+\frac{A^2x^2z^2}{2}+\frac{A^3x^3z^3}{2.3}$, etc. (*R.* 108.)

Whence, by comparing the co-efficients of z,

$Ax=V$, and $x=\frac{V}{A}=\frac{y-\frac{1}{2}y^2+\frac{1}{3}y^3-\frac{1}{4}y^4+\frac{1}{5}y^5}{A}$, etc. $=$

$M.\{y-\frac{y^2}{2}+\frac{y^3}{3}-\frac{y^4}{4}+\frac{y^5}{5}$, etc.$\}$ putting $M=\frac{1}{A}$.

Again, putting $\frac{1}{1-y}=$ a number whose logarithm is required, and making $\frac{1}{1-y}=(1+b)^v$, we find

v (the log. of $(1+b^v)$ or $\frac{1}{1-y}$)$=$

$M\times\{y+\frac{y^2}{2}+\frac{y^3}{3}+\frac{y^4}{4}+\frac{y^5}{5}$, etc.$\}$ by a process exactly analogous to the former.

Now the logarithm of $\frac{1+y}{1-y}$ (or of $\overline{1+y}\times\frac{1}{1-y}$ $=$ logarithm of $(1+y)+$ logarithm of $\frac{1}{1-y}=$

$x+v=2M\{y+\frac{y^3}{3}+\frac{y^5}{5}+\frac{y^7}{7}$, etc.$\}$ (*U.*)

As is manifest by adding together the values of x and v above obtained.

110. As A, and consequently $\frac{1}{A}$ or M, is given in terms of b, (a—1) and a may be assumed of any value whatever, (art. 108,) it follows that M is not limited to one particular value.

If the value of a be assumed, M may be thence determined, or if we take any number at pleasure as the value of M,* a will be limited; but in this case the formula U may be applied to the construction of logarithms, without computing the value of a. M is termed the modulus, and a the radix of the system. Instead of fixing the value of M, or a, by arbitrary assumption, we may take 1 as the logarithm of any proposed number, and thence deduce the value of M.

Since $a^0=1$, the logarithm of 1 must in every system, $=0$. If we make 1= logarithm of 10, the radix will be 10, and M might have been found from equation I, (art. 108,) by assuming $b=9$, if that series or its reciprocal had been a converging one. As that is not the case, a different expedient must be used.

Let $\frac{p+1}{p}$ denote a number whose logarithm is to be found, and put $\frac{1+y}{1-y}=\frac{p+1}{p}$; then $y=\frac{1}{2p+1}$; this value substituted for y in equation U, (art. 109,) will make the series always converge, and more rapidly the greater the value of p.

---

* When M is assumed $=1$, the logarithms thence obtained are termed Naperian, from the name of the inventor, or hyperbolical, from their relation to the areas contained between the curve and asymptote of an equilateral hyperbola.

Now since $\frac{p+1}{p} \times \frac{p}{1} = p+1$, it is evident that the log. of

$\overline{p+1}$ = log. p + log. of $\frac{p+1}{p}$.

First, let $p=1$, then $\frac{p+1}{p}=2$, and $\frac{1}{2p+1}=\frac{1}{3}$; also log. $p=0$.

$$\frac{1}{3}=.3333333333$$

$$\frac{1}{3^3}=\frac{1}{9.3}= \quad 370370370$$

$$(\tfrac{1}{3})^5=\tfrac{1}{9} \text{ of } (\tfrac{1}{3})^3= \quad 41152263$$

$$(\tfrac{1}{3})^7=\tfrac{1}{9} \text{ of } (\tfrac{1}{3})^5= \quad 4572474 \ (A.)$$

$$(\tfrac{1}{3})^9=\tfrac{1}{9} \text{ of } (\tfrac{1}{3})^7= \quad 508053$$

$$(\tfrac{1}{3})^{11}=\tfrac{1}{9} \text{ of } (\tfrac{1}{3})^9= \quad 56450 \ (B.)$$

$$(\tfrac{1}{3})^{13} \quad = \quad 6272 \ (C.)$$

$$(\tfrac{1}{3})^{15} \quad = \quad 697$$

$$(\tfrac{1}{3})^{17}=) \quad 77 \ (D.)$$

If then, we divide these terms by the co-efficients, 1, 3, 5, 7, etc., we shall have—

$$\frac{1}{3} = .3333333333$$

$$\frac{1}{3.3^3} = \quad 123456790$$

$$\frac{1}{5.3^5} = \quad 8230453$$

$$\frac{1}{7.3^7} = \quad 653211$$

$$\frac{1}{9.3^9} = \quad 56450$$

$$\frac{1}{11.3^{11}} = \quad 5132$$

$$\frac{1}{13.3^{13}} = \quad 483$$

$$\frac{1}{15.3^{15}} = \quad 46$$

$$\frac{1}{17.3^{17}} = \quad 4$$

$$.3465735902$$

Hence, log. of $2 = .6931471804M$, and log. of 8 (or 2^3) $= 2.0794415412M$.

Second, let $p=4$, then $\frac{1}{2p+1} = \frac{1}{9}$

Hence to find the log. of $\frac{5}{4}$

$$\frac{1}{9} = .1111111111$$

$$\frac{1}{3} \text{ of } (\frac{1}{9})^3 = A = \quad 4572474$$

$$\frac{1}{5} \text{ of } (\frac{1}{9})^5 = .6B = \quad 33870$$

$$\frac{1}{7} \text{ of } (\frac{1}{9})^7 = \frac{C}{21} = \quad 299$$

$$\frac{1}{9} \text{ of } (\frac{1}{9})^9 = \frac{D}{27} \quad 3$$

$$.1115717757$$

Whence, log. of $\frac{5}{4} = 2231435514M.$

Consequently the log. of $(\frac{8}{1} \times \frac{5}{4})$ 10

$$= 2.3025850926M.$$

But log. of $10 = 1.$

Hence, $M. = \frac{1}{2.3025850926} = .4342944822$

Substituting for M the number last found, we obtain the log. of $2 = .30102999599$, and thence the log. of 4, 8, 16, 32, etc. may be had by simple multiplication. Also the log. of 5= log. of 10— log. of $2 = .69897000401$; whence may be found the logarithms of all the powers of 5.

The numerical value of M being thus ascertained, if we make p denote a number whose log. is known, we have

$$\log \overline{p+1} = \log. p + \log. \frac{p+1}{p} = \log. p +$$

$$\frac{2M}{2p+1} + \frac{A}{3.(2p+1)^2} + \frac{B}{5.(2p+1)^3} + \frac{C}{7.(2p+1)^2}, \text{ etc.}$$

Where A, B, C, etc. denote the preceding term exclusive of the divisor, 3, 5, etc.; and thus the logarithms of all the prime numbers may be computed. But as the above series, when the number p is small, converges but slowly, and therefore requires a considerable number of terms to be used, the labor may be abridged by proper expedients, a few of which are subjoined.

Let the log. of 3 be required; put $p=80$, then

$$\frac{1}{2p+1}=\frac{1}{161}$$

And log. of $\frac{p+1}{p}=\frac{2M}{161}+\frac{A}{3\times 25921}=$ .0053950319

To which add log. of $80=*1+3$ log. of $2=$ 1.9030899880

Hence, log. of 81, or 4 log. of 3= 1.9084850199

And log. 3=.4771212549.

Again, let the log. of 7 be required. Put $p=27$, then

$$\frac{1}{2p+1}=\frac{1}{55}$$

And log. $\frac{28}{27}=\frac{2M}{55}+\frac{A}{9075}$ = .0157942671

Adding log. of 3^3, or of 27 = 1.4313637647

Log. of 28= 1.4471580318

Subtracting log of 4, .6020599920

Log. of 7 = .8450980398

* As the logarithms of 10 and its powers are wholly integral, it is manifest the logarithm of a number is changed only in the integral part, by varying the position of the decimal point in the number itself.

In general, suppose $x+2$ to denote a prime number whose logarithm is required, those of the inferior numbers being known.

Since $\frac{(x-1)^2\times(x+2)}{(x+1)^2\times(x-2)}=\frac{x^3-3x+2}{x^3-3x-2}$, where both terms are divisible by 4, because $x-1$, and $x+1$ are even numbers, it follows that this fraction in its lowest terms, may be expressed by $\frac{p+1}{p}$, and therefore the log. of

$$\frac{(x-1)^2.(x+2)}{(x+1)^2.(x-2)}$$

be obtained from the formulæ above given; to which adding 2 log. $(x+1)$+log. $(x-2)$—2 log. $(x-1)$ the result will be the log. of $x+2$.

Let the log. of 13 be required. Here $x=11$, and our fraction becomes

$$\frac{10^2.13}{12^2.9}=\frac{1300}{1296}=\frac{325}{324}$$

Hence, $p=324$, and $\frac{1}{2p+1}=\frac{1}{649}$

Whence, log. $\frac{325}{324}$ = .0013383507. To which add 4 log. 3+log. of 4--2, log of 5=.1126050039

Log. of 13=1.1139433546

In this manner the logarithms may be derived from those already obtained, to any proposed extent. The differential method may be advantageously applied to the completion of a logarithmic table. But the labor of those computations, being already finished. and not likely to be renewed, a further elucidation of the subject is deemed unnecessary.

Section X.

SURDS.

111. It has been remarked (note, art. 38,) that the even roots of negative quantities are impossible; hence whenever, in the solution of a problem, the square root of a negative quantity appears in the result, such result is impossible, or imaginary. But instead of abandoning, as hopeless, every example in which such expressions appear, they are found conformable to the general principles of the science, and sometimes connected with the most refined analytical processes.

Impossible roots may be introduced into a problem in two ways, quite distinct from each other. First, by admitting incompatible assumptions into the data of the problem; in which case the impossible root serves to detect that incompatibility; and its appearance or disappearance marks the limits of the problem.

Thus in the equation $x^2-2ax=-b$, we have (by art. 67,) $x=a\pm\sqrt{a^2-b}$ as the general expression of the value of x. If now $a^2>b$,* a^2-b is positive, and therefore, $\sqrt{a^2-b}$, and consequently x, a possible quantity. But if $a^2<b$, a^2-b is negative; and therefore, $\sqrt{a^2-b}$ impossible; whence x is, in this case, an imaginary quantity. Here the terms of the original equation are compatible, whenever a^2 is equal to, or greater than b; but incompatible when $a^2<b$; for $x^2-2ax=-b$ is equivalent to $\overline{2a-x}.x=b$, but $\overline{2a-x}.x$ cannot be greater than a^2.

Put $x=a\pm c$, then $2a-x=a\mp c$, and $\overline{2a-x}.x=a^2-c^2$.

When $c=0$, $x=a$, or $\overline{2a-x}.x=(b=\backslash a^2\cdot$ this is therefore the limit of b.

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*$a>b$ is read a is greater than b, and $a<b$, and a is less than b.

Suppose $x+\frac{1}{x}=2a$, to find x.

Multiplying by x, $x^2+1=2ax$, or $x^2-2ax=-1$.

Whence, $x=a\pm\sqrt{a^2-1}$, which value of x is imaginary when $a<1$. Hence the least possible sum of a number and its reciprocal is 2.*

112. Impossible roots may be sometimes obtained, when the data are compatible, by the admission of inconsistent suppositions into the solution.

Suppose we have $x^2+ny^2=a$, and $xy=b$, to find x and y; a, b, and n being any given numbers.

To and from the first equation add and subtract $2\sqrt{n}$ times the second, whence, $x^2+2xy\sqrt{n}+ny^2=a+2b\sqrt{n}$, and $x^2-2xy\sqrt{n}+ny^2=a-2b\sqrt{n}$; and by evolution,

$$x+y\sqrt{n}=\sqrt{a+2b\sqrt{n}},\ (\text{M.})$$

$$x-y\sqrt{n}=\sqrt{a-2b\sqrt{n}},\ (\mathcal{N}.)$$

$$\therefore x=\frac{\sqrt{a+2b\sqrt{n}}+\sqrt{a-2b\sqrt{n}}}{2}$$

---

*These principles may be applied to determine the maxima and minima of geometrical quantities. Let b = the base, a = the altitude of a plane triangle, x = altitude of its inscribed rectangle; then $bx-\frac{bx^2}{a}$ = the area of the rectangle, which put $=c$: whence we find $x=\frac{a}{2}\pm\sqrt{(\frac{a^2}{4}-\frac{ac}{b})}$, which is impossible when $c>\frac{ab}{4}$.

If $c=\frac{ab}{4}$, $x=\frac{a}{2}$, this is, therefore, the greatest possible value of x, and the greatest rectangle is $\frac{ab}{4}=\frac{1}{2}$ the triangle.

$$\text{And } y=\frac{\sqrt{a+2b\sqrt{n}}-\sqrt{a-2b\sqrt{n}}}{2\sqrt{n}}$$

Which are general expressions for the values of x and y.

If now we take $n=-1$, or $x^2-y^2=a$, the above expressions become

$$x=\frac{\sqrt{a+2b\sqrt{-1}}+\sqrt{a-2b\sqrt{-1}}}{2}$$

$$y=\frac{\sqrt{a+2b\sqrt{-1}}-\sqrt{a-2b\sqrt{-1}}}{2\sqrt{-1}}$$

These values, though expressed by imaginary quantities, are real ones. For putting -1 instead of n, in equations M, N, and multiplying, we have

$$x^2+y^2=\sqrt{a^2+4b^2}, \ (P.)$$

But from the first equation, $x^2-y^2=a$.

Consequently, by addition and subtraction,

$$2x^2=\sqrt{a^2+4b^2}+a.$$

And $2y^2=\sqrt{a^2+4b^2}-a$,

$$\text{or } x=\left(\frac{\sqrt{a^2+4b^2}+a}{2}\right)^{\frac{1}{2}}; \quad y=\left(\frac{\sqrt{a+4b^2}-a}{2}\right)^{\frac{1}{2}}$$

The same conclusions may be obtained by squaring the values of x and y first found, and extracting the square roots of the results.

The impossible surd $\sqrt{-1}$, was evidently introduced into the result, by adopting a process in the general solution, which was not applicable to the particular equation $x^2-y^2=a$; yet the values of x and y, when cleared of imaginary surds, are the true ones; as may be shown by a different solution of the problem.

To the square of the first equation adding 4 times the square of the second, $x^4+2x^2y^2+y^2=a^2+4b^2$.

Whence, by evolution, $x^2+y^2=\sqrt{a^2+4b^2}$, the same as equation P, deduced from imaginary surds.

113. From what is shown in the foregoing article, we readily infer, that when, in the solution of a problem, the value of the quantity sought appears in terms of imaginary surds, we are not thence immediately to conclude, that the data are inconsistent; as the adoption of an inapplicable process may produce such a result; yet in this case the imaginary surd may be eliminated by the use of proper expedients. When, however, the data are inconsistent, no analytical address can clear the final equation of its impossible quantities.

Imaginary surds differ from real ones in this important particular. Real surds, however complex, admit of an approximation to their value; but imaginary surds admit of no approximation, and must either be eliminated, when practicable, or remain the intractable indications of incongruous assumptions.

The following cases exhibit the most useful applications of algebra to surd quantities.

Case I.

114. *To reduce a rational quantity to the form of a surd.*

Raise the given quantity to the power denoted by the index of the surd, and to this power apply the radical sign or index proposed.

EXAMPLES.

1. Reduce 5 to the form of a square root, and a to that of a 4th root.

$$5=\sqrt{5^2}=\sqrt{25}, \text{ and } a=(a^4)^{\frac{1}{4}}=\sqrt[4]{a^4}.$$

2. Reduce 3 to the form of a cube root.

Result, $\sqrt[3]{27}$.

3. Express $-\frac{1}{3}a$ in the form of a cube root.

Result, $(-\frac{1}{27}a^3)^{\frac{1}{3}}$.

4. Reduce $2\sqrt{5}$ to the form of a square root.

Result, $\sqrt{20}$.

5. Reduce $3\sqrt{2}$ to the form of a 4th root.

Result, $(324)^{\frac{1}{4}}$.

6. Express $a+b$ in the form of a square root.

Result, $\sqrt{a^2+2ab+b^2}$.

7. Express $\frac{1+\sqrt{5}}{2}$ in the form of a quadratic surd.

Result, $\sqrt{\frac{6+2\sqrt{5}}{4}}$.

Case 2.

115. *To reduce radical quantities, having different indices, to other equivalent quantities with a common radical sign.*

Reduce the fractional exponents to a common denominator, involve the given quantities to the powers denoted by their respective numerators; and to the results apply the reciprocal of the common denominator as the common exponent.

EXAMPLES.

1. Reduce $a^{\frac{2}{3}}$, $b^{\frac{3}{4}}$, and $c^{\frac{1}{5}}$ to equivalent quantities, having a common exponent.

$\frac{2}{3}, \frac{3}{4}, \frac{1}{5}$ reduced are equivalent to $\frac{40}{60}, \frac{45}{60}, \frac{12}{60}$

Hence, $a^{\frac{2}{3}}=(a^{40})^{\frac{1}{60}}=a^{\frac{40}{60}}$;

$b^{\frac{3}{4}}=(b^{45})^{\frac{1}{60}}=b^{\frac{45}{60}}$; $c^{\frac{1}{5}}=(c^{12})^{\frac{1}{60}}=c^{\frac{12}{60}}$.

2 Reduce $3^{\frac{1}{2}}$, and $4^{\frac{1}{3}}$ to a common index.

Result, $27^{\frac{1}{6}}$, $16^{\frac{1}{6}}$.

3. Reduce $a^{\frac{2}{5}}$, $b^{\frac{1}{3}}$, to a common radical sign.

Result, $a^{\frac{6}{15}}$, $b^{\frac{5}{15}}$.

4. Reduce $(a+x)^{\frac{1}{3}}$, $(a-x)^{\frac{1}{2}}$ to a common index.

Result, $(a^3+2ax+x^2)^{\frac{1}{6}}$, and $(a^3-3a^2x+3ax^2-x^3)^{\frac{1}{6}}$.

5. Reduce $a^{\frac{1}{4}}$, and $x^{\frac{1}{4}}$ to the common index $\frac{1}{12}$.*

Result, $(a^4)^{\frac{1}{12}}$, $(x^3)^{\frac{1}{12}}$.

6. Reduce $4^{\frac{1}{6}}$, and $5^{\frac{1}{4}}$ to a common index.

Result, $(16)^{\frac{1}{12}}$ and $(125)^{\frac{1}{12}}$.

Case 3

116. *To reduce surds to their most simple terms.*

Resolve the surd, if possible, into two factors, one of which shall be the greatest power it contains. Extract the root, and thereto annex the other factor with its proper radical sign.

EXAMPLES.

1. Reduce $\sqrt[3]{250}$ to its simplest terms.

$250=125\times 2=5^3\times 2$. $\therefore \sqrt[3]{250}=5\sqrt[3]{2}$.

2. Reduce $\sqrt{32}$ to its simplest terms.

Result, $4\sqrt{2}$.

---

* When the common index, to which the fractional exponents are to be reduced, is given, divide each of the given exponents by that common index, and involve the given quantities to the powers indicated by the quotients.

3. Reduce $\sqrt[4]{243}$ to its simplest form.

Result, $3\sqrt[4]{3}$.

4. Reduce $\sqrt[3]{5184}$ to its simplest terms.

Result, $12\sqrt[3]{3}$.

5. Reduce $(144)^{\frac{1}{4}}$ to its simplest terms.

Result, $2\sqrt{3}$.

6. Reduce $(a^3b-a^3x)^{\frac{1}{3}}$ to its most simple terms.

Result, $a\sqrt[3]{b-x}$.

7. Reduce $\sqrt{\frac{18}{125}}$ to its simplest terms.*

Result, $\frac{3}{25}\sqrt{10}$.

8. Reduce $\sqrt[3]{\frac{8}{9}}$ to its simplest terms.

Result, $\frac{2}{3}\sqrt[3]{3}$.

9. Reduce $\frac{\sqrt{20}+\sqrt{12}}{\sqrt{5}-\sqrt{3}}$ to its simplest form.

Result, $8+2\sqrt{15}$.

CASE 4.

117. *To add or subtract surd quantities.*

Reduce the quantities, when fractional, to a common denominator; and when the exponents are different, to a common radical sign. Also express the surds in their simplest terms. If then the reduced surds are alike, they may be added or subtracted as other algebraic quantities.

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* When the given surd is fractional, the denominator may generally be made rational, without changing the value of the fraction, by multiplying both terms by proper numbers or quantities. When the denominator consists of two quadratic surds connected by the sign + or −, the proper multiplier consists of the same surds connected by the opposite sign.

When the surds are unlike, the operation must be merely indicated.

EXAMPLES.

1. Required the sum and difference of $\sqrt[3]{128}$ and $\sqrt[3]{16}$.

$\sqrt[3]{128}=4\sqrt[3]{2}$, $\sqrt[3]{16}=2\sqrt[3]{2}$, $4\sqrt[3]{2}+2\sqrt[3]{2}=6\sqrt[3]{2}$,

$4\sqrt[3]{2}-2\sqrt[3]{2}=2\sqrt[3]{2}$.

2. What is the sum of $\sqrt{27}$ and $\sqrt{48}$?

Ans. $7\sqrt{3}$.

3. What is the difference between $\sqrt{50}$ and $\sqrt{18}$?

Ans. $2\sqrt{2}$.

4. Required the sum of $\sqrt[3]{56}$ and $\sqrt[3]{189}$.

Ans. $5\sqrt[3]{7}$.

5. What is the sum of $3\sqrt{a^2b}$ and $5\sqrt{16a^4b}$?

Ans. $(3a+20a^2)\sqrt{b}$.

6. What difference is there between $\sqrt{24a^3b^3}$ and $\sqrt{54ab^4}$? Ans. $(2ab-3b^2)\sqrt{6a}$.

7. Required the sum of $\sqrt[3]{\frac{1}{4}}$ and $\sqrt[3]{\frac{1}{32}}$?

Ans. $\frac{3}{4}\sqrt[3]{2}$.

8. Required the difference between $\frac{\sqrt{14}+\sqrt{12}}{\sqrt{7}-\sqrt{6}}$ and $\frac{\sqrt{6}-\sqrt{4}}{\sqrt{3}+\sqrt{2}}$? Ans. $8\sqrt{2}+4\sqrt{21}+4\sqrt{3}$.

CASE 5.

118. *To multiply or divide surd quantities.*

Like quantities, with different exponents, are multiplied or divided by adding or subtracting their indices.

When the quantities are different reduce them, if necessary, to their equivalent ones with a common radical sign. Then the product or quotient, with a common radical sign applied, will be the quantity sought.*

EXAMPLES.

1. Required the product of $5\sqrt[3]{2}$ by $7\sqrt{3}$.

$5\sqrt[3]{2}=5\sqrt[6]{4}$, $7\sqrt{3}=7\sqrt[6]{27}$, $5\sqrt[6]{4}\times 7\sqrt[6]{27}=35\sqrt[6]{108}$.

2. Divide $\sqrt{21}+\sqrt{15}$ by $\sqrt{7}-\sqrt{5}$.

$$\frac{\sqrt{21}+\sqrt{15}}{\sqrt{7}-\sqrt{5}}=\frac{\sqrt{21}+\sqrt{15}}{\sqrt{7}-\sqrt{5}}\times\frac{\sqrt{7}+\sqrt{5}}{\sqrt{7}+\sqrt{5}}=\frac{12\sqrt{3}+2\sqrt{105}}{7-5}$$

$=6\sqrt{3}+\sqrt{105}$, the quotient required.

3. Required the product of $5\sqrt{2}$ by $7\sqrt{3}$.

Result, $35\sqrt{6}$.

4. What is the product of $3\sqrt{5}$ by $4\sqrt[4]{25}$?

Ans, 60.

5. What is the product of $\frac{5}{3}\sqrt{\frac{3}{8}}$ by $\frac{9}{10}\sqrt{\frac{2}{5}}$?

Ans. $\frac{3}{20}\sqrt{15}$.

6. What is the product of $2\sqrt[3]{14}$ and $3\sqrt[3]{4}$?

Ans. $12\sqrt[3]{7}$.

7. What is the quotient of $6\sqrt[3]{972}$ by $3\sqrt[3]{2}$?

Ans. $6\sqrt[3]{18}$.

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*In the division of surds, as well as of rational quantities, it is frequently convenient to set down the terms as a vulgar fraction, and to reduce that fraction to its simplest form.

8. What quotient will $\frac{5}{6}\sqrt{\frac{1}{50}}$ divided by $\frac{3}{4}\sqrt{\frac{2}{5}}$ make?

Ans. $\frac{1}{9}\sqrt{5}$.

9. What is the product of $(a+b)^{\frac{1}{3}}$ and $(a+b)^{\frac{3}{4}}$?

Ans. $(a+b)^{\frac{13}{12}}$.

10. What is the product of $(a+2\sqrt{b})^{\frac{1}{2}}$ and $(a-2\sqrt{b})^{\frac{1}{2}}$?

Ans. $(a^2-4b)^{\frac{1}{2}}$.

11. What is the product of $3+\sqrt{-2}$ by $5-\sqrt{-2}$?

Ans. $17+2\sqrt{-2}$

12. What is the product of $(a+\sqrt{-b})^{\frac{1}{2}}$ by $(a-2\sqrt{-b})^{\frac{1}{2}}$?

Ans. $(a^2+2b-a\sqrt{-b})^{\frac{1}{2}}$.

13. Divide $\frac{4}{5}\sqrt{a}$ by $\frac{2}{3}\sqrt[3]{a}$. Quotient, $\frac{6}{5}a^{\frac{1}{6}}$.

14. Divide $3a^{\frac{1}{n}}$ by $4a^{\frac{1}{m}}$ Quot. $\frac{3}{4}a^{\frac{m-n}{mn}}$.

15. Divide $5+3\sqrt{-3}$ by $7-2\sqrt{-3}$.

Quot. $\frac{17}{61}+\frac{31}{61}\sqrt{-3}$.

As involution is effected by the multiplication of equal factors, this case evidently includes the involution of surd quantities.

16. What is the 3d power of $4\sqrt{2}$?

Ans. $128\sqrt{2}$.

17. Required the nth power of $a^{\frac{1}{m}}$? Ans. $a^{\frac{n}{m}}$.

18. What is the 3d power of $3+2\sqrt{-1}$?

Ans. $46\sqrt{-1}-9$.

Case 6.

119. *To extract the roots of surd quantities.*

When the quantity consists of one term only, the root is obtained by dividing the index of each quantity by the index of the power.

When the quantity is a compound one, the root may sometimes be extracted, as in article 39.

When the given quantity is a binomial, one term, at least, of which is a quadratic surd, the square root may sometimes be extracted by the following formula.

Let $\sqrt{a \pm \sqrt{b}} = x$, put $\sqrt{a^2 - b} = d$, then

$$x = \sqrt{(\tfrac{1}{2}a + \tfrac{1}{2}d)} \pm \sqrt{(\tfrac{1}{2}a - \tfrac{1}{2}d)}^*$$

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* This formula is thus obtained.

Let $\sqrt{a \pm \sqrt{b}} = \sqrt{v} \pm \sqrt{w}$.

Then $a \pm \sqrt{b} = v \pm 2\sqrt{vw} + w$.

Now, as a rational quantity cannot be equal to a surd, we must take

$$a = v + w, \ (N.) \text{ and } \sqrt{b} = 2\sqrt{vw}.$$

By squaring these equations,

$$a^2 = v^2 + 2vw + w^2, \text{ and } b = 4vw.$$

Whence, by subtraction and evolution,

$$\sqrt{a^2 - b} = \sqrt{v^2 - 2vw + w^2}, \text{ or } d = v - w.$$

From this and equation N, we have

$$v = \frac{1}{2}a + \frac{1}{2}d, \text{ and } w = \frac{1}{2}a - \frac{1}{2}d.$$

$$x = \sqrt{v} \pm \sqrt{w} = \sqrt{(\tfrac{1}{2}a + \tfrac{1}{2}d)} \pm \sqrt{(\tfrac{1}{2}a - \tfrac{1}{2}d.)}$$

Hence, it appears that this method applies only to the case where $a^2 - b$ is a complete square.

EXAMPLES.

1. Required the square root of $16\sqrt{6}$.

$\sqrt{16}=4$, and $\sqrt{6^{\frac{1}{2}}}=6^{\frac{1}{4}}$. $\therefore \sqrt{16\sqrt{6}}=4\sqrt[4]{6}$.

2. What is the square root of 6^3?

Ans. $6^{\frac{3}{2}}$ or $6\sqrt{6}$.

3. What is the cube root of $\frac{1}{27}\sqrt{3}$?

Ans. $\frac{1}{3}\sqrt[6]{3}$

4. What is the square root of $a^2-6a\sqrt{-b}-9b$?

Ans. $a-3\sqrt{-b}$.

5. Required the square root of $6+2\sqrt{5}$.

Here $a=6$, $\sqrt{b}=2\sqrt{5}=\sqrt{20}$.

$\therefore b=20$, and $d=\sqrt{36-20}=4$.

Whence $(6+2\sqrt{5})^{\frac{1}{2}}=\sqrt{\frac{6+4}{2}}+\sqrt{\frac{6-4}{2}}=\sqrt{5}+1$, the root required.

6. What is the square root of $3-2\sqrt{2}$?

Ans. $\sqrt{2}-1$.

7. What is the square root of $7-2\sqrt{10}$?

Ans. $\sqrt{5}-\sqrt{2}$.

PROMISCUOUS EXAMPLES.

120. Required the difference between

$\frac{\sqrt{5+2\sqrt{5}}}{2}$ and $\frac{3+\sqrt{5}}{2\sqrt{(5+2\sqrt{5})}}$; $\frac{\sqrt{5+2\sqrt{5}}}{2}=$

$\frac{5+2\sqrt{5}}{2\sqrt{(5+2\sqrt{5})}}$; $\frac{5+2\sqrt{5}}{2\sqrt{(5+2\sqrt{5})}}-\frac{3+\sqrt{5}}{2\sqrt{(5+2\sqrt{5})}}=$

$\frac{2+\sqrt{5}}{2\sqrt{(5+2\sqrt{5})}}=\frac{\sqrt{9+4\sqrt{5}}}{2\sqrt{(5+2\sqrt{5})}}=$ (multiplying num. and denom. by $\sqrt{5-2\sqrt{5}}$) $\frac{\sqrt{5+2\sqrt{5}}}{2\sqrt{5}}=\frac{1}{2}\sqrt{(1+\frac{2}{5}\sqrt{5})}$ the difference sought.

2. Required the sum of $5\sqrt{2}-1$, and $\frac{3\sqrt{2}+2}{5-3\sqrt{2}}$?

Ans. $8\sqrt{2}+3$.

3. What is the sum of $12\sqrt[3]{\frac{1}{4}}$ and $3\sqrt[3]{\frac{1}{32}}$?

Ans. $\frac{27}{4}\sqrt{2}$.

4. Required the difference of $3\sqrt[3]{\frac{1}{3}}$ and $\sqrt[3]{72}$?

Ans. $\sqrt{9}$.

5. What difference is there between $2\sqrt{a^6b^4}$ and $\sqrt{8a^3b^7}$?

Ans. $2ab(a-b)\sqrt[3]{b}$.

6. Required the product of $4+2\sqrt{2}$ by $2-\sqrt{2}$?

Ans. 4.

7. Multiply $a+b\sqrt{-1}$ by $a-b\sqrt{-1}$.

Product, a^2+b^2.

8. Divide $\frac{4+\sqrt{-3}}{2}$ by $\frac{1-\sqrt{-3}}{2}$.

Quotient, $\frac{1+5\sqrt{-3}}{4}$.

9. Required the square root of $51-10\sqrt{2}$?

Ans. $5\sqrt{2}-1$.

10. Required the 3d power of $a-b\sqrt{-1}$?

Ans. $x^3-3a^2b\sqrt{-1}-3ab^2+b^3\sqrt{-1}$.

11. Required the square root of $7-24\sqrt{-1}$?

Ans. $4-3\sqrt{-1}$.

12. Divide a^2+b^2 by $a-b\sqrt{-1}$.

Quotient, $a+b\sqrt{-1}$.

13. Add $\frac{a-b\sqrt{-1}}{a+b\sqrt{-1}}$ to $\frac{a+b\sqrt{-1}}{a-b\sqrt{-1}}$.

Sum, $\frac{2a^2-2b^2}{a^2+b^2}$.

14. Add $\frac{\sqrt{7}}{\sqrt{7}+\sqrt{3}}$ and $\frac{\sqrt{3}}{\sqrt{7}-\sqrt{3}}$ together.

Sum, $\frac{5}{2}$.

15. Divide $18+26\sqrt{-1}$ by $3+\sqrt{-1}$.

Quot. $8+6\sqrt{-1}$.

16. Required the square root of $13-20\sqrt{-3}$?

Ans. $5-2\sqrt{-3}$.

Imaginary surds are of great importance, in the investigation of several valuable formulæ, in the arithmetic of sines.

Section XI.

EQUATIONS IN GENERAL.

121. It has been remarked, page 90, that quadratic equations sometimes admit of more answers than one. The principles on which the ambiguity of quadratic equations depends, are productive of similar results in equations of the higher orders.

In the general equation $x^n - px^{n-1} + qx^{n-2} \ldots\ldots \pm r = 0$, where p, q, r, are supposed to be given, the different values, of which x is susceptible, are called the roots of the equation.

122. If two binomials, as $x-a$, $x-b$, be multiplied together, the product $x^2 - ax - bx + ab$, or $x^2 - (a+b)x + ab$, is manifestly a quadratic; if now, $x-a=0$, or $x-b=0$, we have evidently $x^2 - (a+b)x + ab = 0$. On the other hand, the given quadratic equation $x^2 - px + q = 0$, may be resolved by making $a+b=p$, $ab=q$, and determining a and b. For, on these assumptions being made, the equations become identical, and their conditions are fulfilled by taking $x=a$, or $x=b$; the equation $x^2 - px + q = 0$, has, therefore, two positive roots, which are both possible when $\frac{1}{4}p^2$ is greater than q, and both impossible when q exceeds $\frac{1}{4}p^2$. This expression corresponds to form 3d, $\mathcal{A}$, (art. 67.)

123. Again, $\overline{x-a} \times \overline{x+b} = x^2 - ax + bx - ab$, which being supposed $=0$, corresponds to $x^2 + px - q = 0$, or to $x^2 - px - q = 0$, according as b is greater or less than a.

The conditions of this equation are answered by taking $x-a=0$, or $x+b=0$; the equation has, therefore, two roots, a, and $-b$. These expressions correspond to forms 1, 2, $\mathcal{A}$, (art. 67.) These equations consequently admit of a negative and a positive root, which are both possible; because when the square is completed, the second member consists of positive quantities.

124. Moreover, $(x+a).(x+b) = x^2 + ax + bx + ab$, which being supposed $=0$, agrees with $x^2 + px + q = 0$.

Hence an equation of this form is resolved by making $a+b=p$, $ab=q$, and determining a and b. The roots of this equation are manifestly both negative; and (as in art. 122,) both possible or both impossible.

125. It is worthy of remark, that in the equation $x^2 - px + q = 0$, the signs are alternately $+$ and $-$, that is, they are *twice* changed; and the equation has *two* positive roots. But in the equations (art. 123,) the same sign is

once continued, and the sign *once* changed; and these equations have *one* affirmative and *one* negative root. In the equation (art. 124,) the sign is *twice* continued; and this equation has *two* negative roots.

It also appears, from what is above shown, that every quadratic equation has two roots, which are both possible, or both impossible.*

126. Assuming three factors, $(x-a).(x-b).(x-c)= x^3-(a+b+c)x^2+(ab+ac+bc)x-abc$, which being supposed $=0$, will be identical with $x^3-px^2+qx-r=0$, if $p=a+b+c$, $q=ab+ac+bc$, and $r=abc$. But the conditions of the former are answered by making $x-a=0$, $x-b=0$, or $x-c=0$; therefore, the latter has three positive roots, a, b, and c. It has likewise *three* changes of the signs.

127. If instead of $x-c$ we take $x+c$, our equation $(x-a).(x-b).(x+c)=x^3-(a+b-c)x^2+(ab-ac-bc)x+abc=0$, will, manifestly, have *two positive* roots, a and b. and one negative root, $-c$. In this case, if $a+b>c$, the sign of the second term is $-$, the equation may, therefore, be expressed $x^3-px^2\pm qx+r=0$, in which there are *two* changes of the signs, and one continuation of the same sign. If $c>a+b$, the second term will have the sign $+$, but the third $-$, because in that case $(a+b)$. $c>(a+b)^2$, and therefore, $>ab$. The equation then becomes $x^3+px^2-qx+r=0$; having as before *two* changes of the signs, and *one* continuation of the same sign.

128. A cubic equation, composed of the factors $(x-a).(x+b).(x+c)=x^3+(b+c-a)x^2-(ab+ac-bc)x-abc=0$, has plainly one positive root, a, and two negative roots, $-b$, and $-c$. But in this case, if $b+c>a$, the sign of the second term is $+$; and the equation may be expressed $x^3+px^2\pm qx-r=0$, having *one* change of the signs, and

* In form 3, *A*, (art. 67,) when $b=a^2$, the roots are equal to each other. In forms 1 and 2, if $2a=0$, the equation becomes a simple quadratic, and the roots are equal quantities with contrary signs. Every simple quadratic has, therefore, a positive and negative root.

two continuations of the same sign. If $a>b+c$, the second and third terms are both negative, because $(b+c)a>(b+c)^2>bc$; the equation, therefore, may be expressed $x^3-px^2-qx-r=0$, having, as before, *one* change of the signs, and *two* continuations of the same sign.

129. If we use the factors $x+a$, $x+b$, $x+c$, their product $x^3+(a+b+c)x^2+(ab+ac+bc)x+abc$, being put $=0$, the equation may be expressed, $x^3+px^2+qx+r=0$, in which there are three continuations of the same sign. The equation has likewise *three* negative roots, $-a$, $-b$, and $-c$.

By pursuing this inquiry it will be found, that any equation of this kind admits of as many roots as there are units in the index of the highest power of the unknown quantity; that the number of positive and negative roots will be, respectively, equal to the number of changes in the signs, and the number of continuations of the same sign. It likewise appears, that the last term, or absolute number, is the continued product of all the roots with their signs changed.

130. It may be observed, that as every cubic equation is composed of three factors, and every quadratic of two; a cubic equation may always be considered as the product of a simple and a quadratic equation. But (art. 125,) every quadratic has, either two possible, or two impossible roots; hence, a cubic equation, the terms of which are possible, having *one* impossible root, has *two;* and, as in the multiplication of compound quantities, containing impossible parts, those impossible parts can disappear only when two like roots are multiplied together, it follows, that every cubic equation, consisting of possible quantities, has, at least, one possible root.

131. In like manner, it appears that every equation consisting of possible quantities, having an odd number of roots, has, at least, one of those roots possible. And that

every equation which is made up of possible quantities, has all its roots possible, or an *even* number of impossible roots.

132. When the roots of an equation are integral, they may sometimes be found with great facility, by seeking the divisors of the last term, and substituting them in place of the unknown quantity, till one or more are found which answer the conditions of the equation. (See art. 129.)

When one root has been found, the equation may be depressed by connecting that root, with its sign changed, with the unknown quantity, and dividing the given equation by the sum.

EXAMPLES.

1. Given $x^3-3x^2+5x-15=0$, to find the value of x.

(Art. 126,) the roots, if possible, are all positive.*

Also, the divisors of 15 are 1, 3, 5, 15.

Now, by substituting these for x,

$$1-3+5-15=-12. \quad \therefore 1 \text{ is not a root.}$$

$$27-27+15-15=0. \quad \therefore 3 \text{ is a root.}$$

$x-3=0$, and $\dfrac{x^3-3x^2+5x-15}{x-3}=x^2+5=0.$

Whence, $x=\pm\sqrt{-5}$.

$\therefore$ the roots are 3, $+\sqrt{-5}$, and $-\sqrt{-5}$.

2. Given $x^3-2x^2-5x+6=0$, to find x.

Results, 1, 3, -2.

---

* The rules, in the foregoing articles, for determining the signs of the roots from the changes in the signs of the terms composing the equation, being founded on the supposition that each root has but one sign, do not apply to impossible roots; because the negative signs under the radicals, when developed by multiplication, are combined with those of the roots, and, therefore, change the signs of the terms of which the equation is composed.

3. Given $x^3+6x^2-7x-60=0$, to find x

Results, 3, —4, —5

4. Given $x^3+3x^2-6x-8=0$, to find x.

Results, 2, —1, —4.

5. Given $x^3-2x+4=0$, to find x.

Results, -2, $1+\sqrt{-1}$, $1-\sqrt{-1}$.

6. Given $x^4-10x^3+35x^2-50x+24=0$, to find x.

Results, 1, 2, 3, 4.

7. Given $x^4-8x^3+23x^2-64x+120=0$, to find x.

Results, 5, 3, $2\sqrt{-2}$, $-2\sqrt{-2}$.

133. When the roots are not integral, they may generally be determined by approximation.* For this purpose, various rules have been investigated. Among these the following is probably the most convenient in practice. The demonstration is given in the subsequent article.

To find the root of a general equation.

1. If all the terms of the equation are not on one side, by transposition, place them so; and arrange them according to the powers of the unknown quantity, placing the highest power on the left hand. If any of the lower powers are not contained in the equation, consider each one omitted, as having a *cipher* for its *co-efficient.*

2. Place the co-efficients and the absolute number with their proper signs, in order, in a horizontal line.

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*If all the roots of the equation are impossible, this method is not applicable, (see art. 111,) but possible roots may always be approximated.

3. Find by trial the first root figure, attending to its value as being units, tens, tenths, or hundredths, etc and place it to the right of the absolute number.

4. Multiply the first co-efficient by the root figure, and add the product to the second co-efficient; multiply the sum by the root figure, and add the product to the third co-efficient; proceed thus to the end of the line, adding the last product to the absolute number. Again, multiply the first co-efficient by the root figure, and add the product to the sum under the second co-efficient; multiply the resulting sum by the root figure, and add the product to the sum under the third co-efficient; and so on, stopping under the last co-efficient. Repeat the process, stopping each succeeding time, one term nearer to the left hand, till the last sum falls under the second co-efficient.

5. Try how often the last sum under the last co-effieient is contained in the sum under the absolute number, and take the result for the next root figure.

6. Using the first co-efficient, and the last sum in each column, instead of the co-efficients and absolute number, proceed with this new root figure as with the preceding one.

7. Obtain another root figure in the manner last mentioned, and thus continue the operations as far as necessary.

Note 1.—In multiplying by each root figure, attention must be given to its value. Thus, if it is of the order of tens, the multiplication must be made by the number of tens which it represents; and so, for other values. Also, in the multiplications and additions, regard must be had to the signs of the numbers.

2. The signs of the successive sums under the absolute number and last co-efficient, must continue the same throughout the operation, or both change by the same root figure.

If the operation for any of the root figures causes only one sign to change, another value must be taken. This will not unfrequently occur with regard to the second root figure, but it will seldom be the case for the others.

3. After two or three root figures have been obtained, and the multiplications and additions corresponding to them have been completed, the succeeding parts of the operation may be contracted in the following manner. Cut off the right hand figure of the sum, in the column under the last co-efficient; the two right hand figures of the sum in the preceding column; the three right hand figures of the sum in the column preceding that, and so on.

If either of the figures next to the right of the marks of separation is 5, or more than 5, add, mentally, a unit to the first figure on the left of the mark, when using it in the succeeding multiplication and addition. Repeat the same contraction for each of the following root figures.

These contractions may commence, in cubic equations, after the second or third decimal figure in the root is obtained; in biquadratic equations, after the first or second decimal figure; and in higher equations, after the first decimal figure. And if the operation is closed when the sum under the absolute number is reduced to two figures, all the figures in the root will be true.*

EXAMPLES.

1. Given $3x^4 - 4x^3 + 2x - 1000 = 0$, to find the value of x. Ans. 4.342447603.

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*This rule, improved from Young's Algebra, was communicated by my friend John Gummere, of Burlington.

3	0	—4	+2	—1000(4.342447603
	12	48	176	712
	12	44	178	—288
	12	96	560	248.2803
	24	140	738	—39.7197
	12	144	89.601	37.39678208
	36	284	827.601	—2.32291792
	12	14.67	94.083	1.89781666
	48.9	298.67	921.684	—.42510126
	.9	14.94	13.235552	37988512
	49.8	313.61	934.919552	—4521614
	.9	15.21	13.318496	3799436
	50.7	328.82	948.23804,8	—722178
	.9	2.0688	.67028	664909
	51.6	330.8888	948.90833	—57269
	.12	2.0736	.67048	56994
	51.72	332.9624	949.5788,1	—275
	12	2.0784	.1340	285
	51.84	335.04,08	949.7128	
	12	10	1340	
	51.96	335.14	949.846,8	
	.12	10	12	
	5,2.08	335.24	949.859	
		10	12	
		3,35.34	94,9.8,7,1	

2. Given $x^5+2x^4+3x^3+4x^2+5x-54321=0$, to find the value of x. Ans. 8.41445475.

1	2	3	4	5	—54321(8.41445475
	8	80	664	3344	42792
	10	83	668	5349	—11529
	8	144	1816	19872	11088.97344
	18	227	2484	25221	—440.02656
	8	208	3480	2501.4336	304.11052
	26	435	5964	27722.4336	—135.91604
	8	272	289.584	2620.0064	122.02904
	34	707	6253.584	30342.440,0	—13.88700
	8	16.96	296.432	68.612	12.21504
	42.4	723.96	6550.016	30411.052	—1.67196
	.4	17.12	303.344	68.690	1.52700
	42.8	741.08	6853.3,60	30479.74,2	—14496
	.4	17.28	7.8	27.52	12216
	43.2	758.36	6861.2	30507.26	—2280
	.4	17.44	7.8	27.52	2135
	43.6	77,5.80	6869.0	30534.7,8	—145
	.4		7.8	2.8	153
	44.0		6,87,6.8	30537.6	
				2.8	
				30,5,4,0.4	

3. Given $x^3+9x^2+4x=80$, to find x.

Result, $x=2.4721359$.

4. Given $x^3+x^2+x=90$, required the value of x.

Ans. $x=4.1028323$.

5. Given $x^3+10x^2+5x=2600$, required the value of x.

Ans. 11.00679934.

6. Given $2x^4+16x^3+40x^2+30x=4500$, required the value of x. Ans. $x=5.03770809$.

134. To show the rationale of the process directed in the last article, I begin with the cubic equation

$$ax^3+bx^2+cx+d=0.$$

Let the first figure in the root be indicated by r, regard being paid to its *local value*, and the remaining part of the root by y; then $x=y+r$.

$$\text{Hence, } \left.\begin{array}{rrrr} ax^3= & ay^3+ & 3ary^2+ & 3ar^2y+ar^3 \\ bx^3= & & by^2+ & 2bry+br^2 \\ cx= & & & cy+cr \\ d= & & & d \end{array}\right\}=0.$$

Collecting the co-efficients of like powers of y,

$$ay^3+b'y^2+c'y+d'=0.$$

It is obvious that $d'=\{(ar+b)r+c\}r+d$;

$c'=(ar+b)r+(2ar+b)r+c$; $b'=ar+ar+ar+b$.

But these are the quantities found by the fourth precept.

Again, since $r>y$, $c'y+d'$ approximates to 0, or $y=\frac{-d'}{c'}$ nearly; but this is the mode prescribed in the fifth precept for finding the next figure of the root.

Denoting the number obtained by the last operation by s, and the remaining part of y by z, so that $y=z+s$, we shall obtain a new equation $az^3+b''z^2+c''z+d''=0$, in which $d'=\{(as+b')s+c'\}s+d'$; $c''=(as+b')s+(2as+b')s+c'$; $b''=as+as+as+b'$; whence an approximate value of z, or a new figure of the root, is manifestly deducible from this new equation, as before.

Assuming now the general equation

$$ax^n+bx^{n-1}\ldots\ldots\ mx^2+nx+p=0.$$

And denoting as before, the first figure of the root by r, and making $x=y+r$, we have

$$\left.\begin{array}{lllll} ax^n=ay^n+nary^{n-1}\ldots & \ldots\text{etc.} & nar^{n-1}y+ar^n & \\ bx^{n-1}= \quad by^{n-1}\ldots & \text{etc.} & \overline{n-1}.br^{n-2}y+br^{n-1} & \\ \text{etc.} \quad \ldots & \text{etc.} & & \\ mx^2= & & 2mry+mr^2 & \\ nx= & & ny+nr & \\ p= & & p & \end{array}\right\}=0.$$

Or $ay^n+b'y^{n-1}+\ldots$ etc. $n'y+p'=0$, in which the quantities b', n', p', are composed of the co-efficients a, b, etc. and the powers of r combined, as directed in the fourth precept. And here as before $y=-\frac{p'}{n'}$ nearly.

Hence it is obvious, that the successive figures of the root may be obtained by the same kind of process; whatever may be the index of the highest power of the unknown quantity.

INDETERMINATE PROBLEMS.

135. When a problem is given, in which the number of unknown quantities employed, is greater than the number of independent equations furnished by the conditions of the problem; one, at least, of those quantities, may be assumed at pleasure. Such problems, therefore, generally admit of an indefinite number of answers. There are, however, certain conditions sometimes annexed, by which the number of answers is partially limited. For example, the answers are required to be whole positive numbers, or they are required to be square or cube numbers. Such problems are termed indeterminate, or unlimited, though, in some instances, each unknown quantity admits of but one value. If simple powers only of the unknown quan-

tities are included in the equations, the problem is said to be of the first degree.

CASE I.

*To find the values of x and y, in whole positive numbers, from the equation $ax=by+c$; a, b, c, being given numbers, positive or negative.**

Here $x=\frac{by+c}{a}$, and as x is to be a whole number, $\frac{by+c}{a}$ must also be a whole number.

Now, if this quantity be multiplied by a whole number, the product must, evidently, be a whole number; also the sum or difference of this quantity, or either of its multiples, and any whole number, must necessarily be a whole number. Let, therefore, $\frac{by+c}{a}$ be thus changed till we obtain $\frac{y+c'}{a}=wh$,† which put $=p$; then $y=ap-c'$, a quantity that must be a whole number, because a, p, and c' are whole numbers. And this value being substituted for y in the equation $x=\frac{by+c}{a}$, the value of x will be obtained in terms of p, and given numbers.

Assuming then $p=0$, 1, 2, 3, etc. successively, (omitting such numbers as make x or y negative,) the various numerical values of x and y become known.

The number of answers will be limited when the signs of p in the values of x and y are unlike: but unlimited when p has the same sign in both.

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* If a and b have a common divisor, it must also be a divisor of c, or the problem is impossible.

† This expression is used to designate any *whole number*.

EXAMPLES.

1. Given $19x=14y+15$, to find the values of x and y in whole positive numbers.

$$x=\frac{14y+15}{19}=wh. \quad \therefore \frac{14y+15}{19}\times 4=$$

$$\frac{56y+60}{19}=2y+3+\frac{18y+3}{19}=wh.$$

But $\frac{19y}{19}=wh. \quad \therefore \frac{19y}{19}-\frac{18y+3}{19}=\frac{y-3}{19}=wh=p$

$$\therefore y=19p+3, \text{ and } x=\frac{\overline{19p+3}\times 14+15}{19}=14p+3.$$

If now, we assume p successively $=0, 1, 2, 3$, etc.

$$y=3, 22, 41, 60, \text{ etc.}$$

$$x=3, 17, 31, 45, \text{ etc.}$$

Here the number of answers is evidently unlimited.

2. Given $11x+17y=987$, to find x and y in whole positive numbers.

$$x=\frac{987-17y}{11}=89-y+\frac{8-6y}{11}=wh. \quad \therefore \frac{8-6y}{11}=wh,$$

$$\frac{8-6y}{11}\times 2=\frac{16-12y}{11}=1-y+\frac{5-y}{11}=1-y-\frac{y-5}{11}=wh.$$

$$\therefore \frac{y-5}{11}=wh=p.$$

Whence $y=11p+5$,

And $x=\frac{987-(11p+5)\times 17}{11}=82-17p.$

Assuming $p=0, 1, 2, 3, 4,$

$$y=5, 16, 27, 38, 49.$$

$$x=82, 65, 48, 31, 14$$

Which are all the possible values in whole positive numbers.

3. Given $7x+9y=2342$, to find the number of values of x and y in whole positive numbers.

$$x=\frac{2342-9y}{7}=334-y-\frac{2y-4}{7}=wh. \quad \therefore \frac{2y-4}{7}=wh.$$

$$\frac{2y-4}{7}\times 4=\frac{8y-16}{7}=y-2+\frac{y-2}{7}=wh. \quad \therefore \frac{y-2}{7}=wh=p.$$

$$\therefore y=7p+2, \text{ and } x=332-9p.$$

From the first of these expressions we perceive that the least value of p is 0, and from the second, that the greatest value of p is 36. Hence the required number is 37.

4. Given $5x+7y+9z=337$, to find the number of values of x, y, and z, in whole positive numbers.

$$x=\frac{337-7y-9z}{5}=67-y-z+\frac{2-2y-4z}{5}=wh.$$

$$\therefore \frac{2-2y-4z}{5}=wh.$$

$$\text{And } \frac{2-2y-4z}{5}\times 2=\frac{4-4y-8z}{5}=wh.$$

$$\text{But } \frac{5y-5+10z}{5}=wh.$$

$$\therefore \frac{5y-5+10z}{5}+\frac{4-4y-8z}{5}=\frac{y-1+2z}{5}=wh=p.$$

$$\therefore y=5p-2z+1, \; x=66-7p+z.$$

Assuming now z successively equal to 1, 2, 3, etc. we shall have the corresponding values of x and y, the greatest and least values of p, and the number of answers as follows:

z	x	y	p Gr.	p Least.	No. of Ans.
1	$67—7p$	$5p—1$	9	1	9
2	$68—7p$	$5p—3$	9	1	9
3	$69—7p$	$5p—5$	9	2	8
4	$70—7p$	$5p—7$	9	2	8
5	$71—7p$	$5p—9$	10	2	9
6	$72—7p$	$5p—11$	10	3	8
7	$73—7p$	$5p—13$	10	3	8
8	$74—7p$	$5p—15$	10	4	7
9	$75—7p$	$5p—17$	10	4	7
10	$76—7p$	$5p—19$	10	4	7
11	$77—7p$	$5p—21$	10	5	6
etc.	etc.	etc.	etc.	etc.	etc.

Here we may observe, there are seven successive values of z, (beginning with the 5th,) which produce no change in the *greatest value* of p; and there are alternately two and three equal *least values* of p; and this order will evidently continue as long as successive values of z are assumed.

The last three columns of the above table may, therefore, be continued without the former ones.

Gr. val. of p,	9,	9,	9,	9,	10,	10,	10,	10,	10,	10,	10,	11,	11,	11,
Least do.	1	1	2	2	2	3	3	4	4	4	5	5	6	6
No. of Ans.	9	9	8	8	9	8	8	7	7	7	6	7	6	6

Gr. val. of p,	11	11	11	11	12	12	12	12	12	12	12	13	13	13
Least do.	6	7	7	8	8	8	9	9	10	10	10	11	11	12
No. of Ans.	6	5	5	4	5	5	4	4	3	3	3	3	3	2

Gr. val. of p,	13	13	13	13	14	14	14	14
Least,	12	12	13	13	14	14	14	15
No. of Ans.	2	2	1	1	1	1	1	0

The number of answers, or sum of the numbers in the last column, is 169.

Otherwise—

From the equation $z=\frac{337-5x-7y}{9}$, since neither x, nor y can be less than 1, it is obvious that z cannot exceed 36. Then in the equation $x=66-7p+z$, make $z=36$ and $x=102-7p$; where the greatest possible value of $p=14$. Put $p=14$ in the equation $y=5p-2z+1$, and $y=71-2z$. From which we perceive that the greatest value of $z=35$. Assuming then $z=1, 2, 3$, etc. to 35, and taking all the values of p, from 14 to the lowest, which will give a positive value to y; and combining the results five by five, we shall have a series in arithmetical progression, $67+57+\ldots 7=259$, which is the number of whole positive values of y. But some of the corresponding values of x will be 0 or negative.

From the equation $x=66-7p+z$, we find the first four values of z, 1, 2, 3, 4, render $x=0$, or negative if $p>9$. Hence, we have 20 inadmissible values of x. For the next 7 values of z, we have $x=0$, or negative if $p>10$; which give 28 inadmissible values. Proceeding until $z=35$, we have the series $20+28+21+\ldots 7=90$. Hence $259-90=169$, the number required.

4. Given $14x=5y+19$, to find the least possible values of x and y, in whole positive numbers.

Ans. $x=6$, $y=13$.

5. Given $11x+5y=254$, to find all the values of x and y, in whole positive numbers.

Ans. $x=19, 14, 9, 4.$
$y=9, 20, 31, 42.$

6. Given $9x+13y=2000$, required the number of values of x and y. Ans. 17.

7. Required to divide 100 into two parts, so that one of them may be divisible by 7, and the other by 11?

The parts are 56 and 44.

8. Given $17x+19y+21z=400$, required the number of values of x, y, and z. Ans. 10.

9. Required to pay 1000 dollars, in French crowns, and five franc pieces, so that the number of coins used shall be the least possible; what number of each kind will be necessary, and how many ways can that sum be paid in those coins, the French crown being 110 cents, and the five franc piece 93 cents?

Ans. 833 crowns, 90 five franc pieces, and 9 different ways.

10. How many gallons of liquor at 12 cents, 15 cents, and 18 cents per gallon, may be mixed, to compose 300 gallons at 17 cents per gallon?*

Ans. { at 12 cents, 1, 2, 3, etc. to 49.
at 15 cents, 98, 96, 94, etc. to 2.
at 18 cents, 201, 202, etc. to 249.

11. In how many ways can £1053 sterling be paid without using any coins besides guineas and moidores; the guinea being 21*s.* sterling, and the moidore 27?

Ans. 112 ways.

12. A foreigner having a bill of $1 75 to pay at a hotel, offers napoleons in payment, the landlord agrees to receive them on condition that Turkish sequins shall be taken as change; how many pieces must be used, a napoleon being worth $7 25, and a sequin rated at $2 10?

Ans. 35 napoleons, and 120 sequins.

13. Given $7x+9y+23z=9999$; of how many values will x, y, and z admit?

Ans. 34365.

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* When more equations than one are given, one, at least, of the unknown quantities may be eliminated, and the equations reduced to one.

CASE 2.

136. *To find a whole number, which, being divided by any given numbers, shall leave given remainders.*

Denoting the required number by x, the given divisors by a, b, c, etc. and the remainders by f, g, h, etc. we shall have $\frac{x-f}{a}$, $\frac{x-g}{b}$, $\frac{x-h}{c}$, etc. severally equal to whole numbers.

Put $\frac{x-f}{a}=p$, find the value of x, and substitute it for x in the second fraction; reduce this new fraction, as in the former case, so that the co-efficient of p may be a unit, and put the fraction thus obtained $=q$; find the value of p, and thence of x, in terms of q and given numbers. Substitute the value of x in the third equation, and so proceed. The last value of x will be the number sought.

EXAMPLES.

1. Required a number, which, being divided by 7, 9, 10, and 11, shall leave the remainders 5, 4, 7, and 9, respectively.

Let $x=$ the number sought.

Then $\frac{x-5}{7}$, $\frac{x-4}{9}$, $\frac{x-7}{10}$, $\frac{x-9}{11}$, are whole numbers.

$$\frac{x-5}{7}=wh=p. \quad \therefore x=7p+5.$$

Whence, $\frac{7p+5-4}{9}=\frac{7p+1}{9}=wh.$

$$\therefore \frac{7p+1}{9}\times 4=\frac{28p+4}{9}=3p+\frac{p+4}{9}=wh.$$

$$\therefore \frac{p+4}{9} = wh = q, \text{ and } p = 9q - 4. \quad \therefore x = 63q - 23.$$

$$\therefore \frac{63p - 30}{10} = 6q - 3 + \frac{3q}{10} = wh. \quad \therefore \frac{3q}{10} = wh.$$

And consequently.

$$\frac{q}{10}^{*} = wh = r. \quad \therefore q = 10r, \text{ and } x = 630r - 23.$$

$$\therefore \frac{630r - 32}{11} = 57r - 3 + \frac{3r+1}{11} = wh. \quad \therefore \frac{3r+1}{11} = wh.$$

$$\text{And } \frac{3r+1}{11} \times 4 = \frac{12r+4}{11} = r + 1 + \frac{r-7}{11} = wh.$$

$$\therefore \frac{r-7}{11} = wh = s. \quad \therefore r = 11s + 7.$$

And $x = 6930s + 4387 = (\text{if } s = 0)\ 4387.$

Assuming $s = 1$, 2, etc. other values would arise.

2. Required a number, which, being divided by 2, 3, 4, 6, shall leave no remainder; but being divided by 7, the remainder shall be 6?

The least common multiple of 2, 3, 4, 6,† is 12.

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* If a fractional expression $\frac{a}{b} = wh$, and we divide the terms into their primes, it is plain that all the prime numbers contained in the denominator, must be also contained in the numerator; thus, $\frac{3q}{10} = \frac{3.q}{2.5} = wh$, where the primes 2 and 5 not being contained in the 3, must be contained in q, and therefore, $\frac{q}{10} = wh$.

† See Arithemeticai Expositor, Part 1, page 90.

Let, therefore, $12x$= the number sought.

$$\therefore \frac{12x-6}{7} - = wh. \quad \therefore \frac{12x-6}{7} \times 3 = \frac{36x-18}{7} = 5x-2+$$

$$\frac{x-4}{7} = wh; \text{ and } \frac{x-4}{7} = wh = p. \quad \therefore x = 7p+4.$$

And $12x = 84p+48 = 48$, 132, 216, etc.

3. Required the least whole number, which, being divided by 3, 5, 7, and 8, shall leave the remainders 1, 3, 5, and 0, respectively? Ans. 208.

4. Required the least whole number, which, being divided by 9, 10, 11, and 12, shall leave the remainders 4, 5, 6, and 7, respectively? Ans. 1975.

5. What is the least whole number, which, being divided by each of the nine digits, shall leave no remainder; but divided by 17, the remainder shall be 13? Ans. 40320.

6. In what year of the Christian era was the solar cycle 15, the lunar cycle 3, and the Roman indiction 14?* Ans. 1826.

7. Required the least whole number, which, being divided by 2, 3, 4, and 5, shall leave a remainder of 1; but divided by 7, there shall nothing remain? Ans. 301.

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*The solar cycle is a period of 28 years, at the end of which, in case a centurial year has not intervened, the days of the week always return to the same days of the month. To the year of the Christian era add 9, and divide by 28, the remainder will be the number of the cycle.

The lunar cycle is a period of 19 years, at the expiration of which the new and full moons return nearly to the same time of the year. To the year of the Christian era add 1, and divide by 19, the remainder is the lunar cycle.

The Roman indiction is not an astronomic period, it consists of 15 years. To a given year add 3, and divide by 15, the remainder is the number of the indiction.

If, in either of these cases, no remainder occurs, the divisor must be taken as the number of the cycle.

MISCELLANEOUS EXAMPLES.

137.—1. A bankrupt owes A. twice as much as he owes B., and he owes to C. as much as to A. and B. together; the sum to be divided is 600 dollars; how much must each receive? Ans. A. 200, B. 100, C. 300.

2. A. can perform a piece of work in 7 days, and B. in 9 days; in what time would they jointly effect it?
Ans. $3\frac{15}{16}$ days.

3. A laborer being employed on condition, that for every day he worked he should receive 50 cents, and for every day he was idle, he should forfeit 20 cents, finds at the end of 500 days only 89 dollars due; how many days did he work? Ans. 270.

4. There are two numbers in the ratio of 4 to 5, and the sum of their squares is 1476; what are the numbers?
Ans. 24 and 30.

5. A Greek epitaph, designed for the tomb of Diophantus, is said to have stated that he passed one-sixth of his life in childhood; one-twelfth in adolescence; that after one-seventh and five years more had been passed in a married state, he had a son who lived to half his own age, and whom he survived four years; what then was the age of Diophantus? Ans. 84 years.

6. A person's age in years is a number, consisting of two digits; $\frac{4}{7}$ of this number is a mean proportional between these digits; and two years hence his age will be a third proportional to those digits, beginning with the tens; what is the age? Ans. 14 years.

7. Given $x+y+z=9$, $xy+xz+yz=26$, $xyz=24$, to find the values of x, y, and z.* Result, $x=2$, 3, or 4.

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* If we assume the equation $v^3-9v^2+26v-24=0$, then art. 126, the roots of this equation will be the values required; and those roots are easily found by art. 132.

8. By selling a piece of muslin at a certain price per yard, I gained the prime cost of 9 yards, which was just as much per cent. as the number of yards in the piece; what was that number?

Ans. 30 yards.

9. There are four numbers in continued proportion, the sum of the first and last 1728, and the sum of the other two 1152; what are the numbers?

Ans. 192, 384, 768, and 1536.

10. The hypothenuse of a right angled triangle is x^{3x}, and the other sides x^{2x}, and x^{x}; what is the area?

Ans. $\frac{1}{2}\sqrt{(2+\sqrt{5})}$.

11. There is a number, consisting of three digits in arithmetical progression, which number being divided by the sum of the digits, the quotient will be 48, but if from the number 198 be subtracted, the remainder will be expressed by the same digits in an inverted order. Quere the number?

Ans. 432.

12. The sum of the squares of two numbers being multiplied by the quotient arising from the division of the less by the greater, produces 83.2, and the difference of the squares multiplied by the quotient of the greater divided by the less, produces 1920; what are the numbers?

Ans. 20 and 4.

13. A ball falling from the top of a tower, is observed to descend one-fourth of the distance in the last second of the time; required the height of the tower, heavy bodies being known to fall $16\frac{1}{12}$ feet during the first second, and to describe spaces, which, reckoned from the beginning of the fall, are as the squares of the times?

Ans. $(28+16\sqrt{3})16\frac{1}{12}$ feet.

14. What are the values of x and z, from the equations $x^3z+xz^3=546560$, $x^4+z^4=1086992$?

Ans. $x=32$, $z=14$.

15. Given $x+y+z+v=56$, $x^2+y^2+z^2+v^2=910$, $xv+2y^2-z^2=6$, and $z=2y$, to find the values of x, y, z, and v

Result, $x=8$, $y=9$, $z=18$, $v=21$.

16. Required to find a number, which, being any way divided into two unequal parts, the greater part added to the square of the less, shall be equal to the less part added to the square of the greater. Ans. 1.

17. Given $xy=125x+300y$, and $y^2-x^2=90000$, to find x and y by a quadratic.* Result, $x=400$, $y=500$.

18. Given $(x^3+1).(x^3+1).(x+1)=30x^3$, to find the value of x by a quadratic equation.†

Result, $x=\frac{1}{2}(3\pm\sqrt{5})$.

19. The sum of three numbers in harmonical proportion is 26, and their continued product 576; required the numbers by a quadratic equation. Ans. 12, 8, and 6.

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*This is readily solved by quadratics, by finding x in terms of y from the first equation, and substituting the value in the second.

†This question requires expedients not readily found.

By multiplication and transposition.

$$x^6+x^5+x^4-28x^3+x^2+x+1=0, \text{ whence}$$

$$x^3+x^2+x-28+\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}=0$$

But $y=x+\frac{1}{x}$, whence $y^3+y^2-2y-30=0$

Assume $z=y+\frac{1}{3}$; wherefore $z^3-\frac{7}{3}z=\frac{790}{27}=\frac{79}{9}.\frac{10}{3}$

And $z^4-\frac{7}{3}z^2=\frac{79}{9}.\frac{10}{3}z$. Adding $\frac{(10z)^2}{9}$ to each member

$$z^4+\frac{79}{9}z^2=(\frac{10}{3}z)^3+\frac{79}{9}.\frac{10}{3}z$$

Hence $z=\frac{10}{3}$, and $y=3$. Whence $x^2-3x=-1$.

$$\text{and } x=\frac{3\pm\sqrt{5}}{2}$$

20. The sum of two numbers is 152, and the cube root of the square of their difference multiplied by the square root of the cube of the same difference, produces 8192; what are the numbers? Ans. 44 and 108.

21. The sum of two numbers, added to the sum of their squares, is 120, and the product of the same numbers 45; what are they? Ans. 9 and 5.

22. Given $x^3+xy^2=4640y$, and $x^2y-y^3=537.6x$; what are the numerical values of x and y?

Ans. $x=40$, $y=16$.

23. Given $x+y=z$, $z^3-x^3+y^3=470$, $z^3-x^3-y^3=468$, to find x, y, and z. Result, $x=12$, $y=1$, $z=13$.

24. Given $\begin{cases} \overline{x+y}^{3}+x+y=30. \\ x-y=1. \end{cases}$ Required the values of x and y.

Ans. $x=2$, $y=1$.

25. Given $x^4-2x^3+x=a$, to find x by a quadratic equation. Result, $x=\frac{1}{2}+\sqrt{(\frac{3}{4}+\sqrt{a+\frac{1}{4}})}$.

26. Given $x^2+x(y+z)=a$, $y^2+y(x+z)=b$, $z^2+z(x+y)=c$, to find x, y, and z.

Results, $\begin{cases} x=\dfrac{a}{\sqrt{(a+b+c)}} \\ y=\dfrac{b}{\sqrt{(a+b+c)}} \\ z=\dfrac{c}{\sqrt{(a+b+c)}}. \end{cases}$

27. Given $2x^3-x^4-x^2+2x^2y-y^2=2xy=\sqrt{x^5y}-\sqrt{x^5y}$, to find x and y. Result, $x=5$, $y=20$.

28. Required the roots of the equation $4x^4+8x^3-89x^2+28x+49=0$, by quadratics only.

Ans. $1, \dfrac{7}{2}, \dfrac{-13+\sqrt{113}}{4}, \dfrac{-13-\sqrt{113}}{4}$

29. There are three numbers in geometrical progression, whose continued product is 4096, and the sum of the extremes is 68; what are the numbers?

Ans. 4, 16, and 64.

30. There are two numbers expressed by the same two digits, and the difference of their squares is 1485; quere the numbers? Ans. 14 and 41.
Or 78 and 87.

31. What two numbers are those whose product, difference of their squares, and quotient of their cubes, are all equal? Ans. $\frac{3}{2}+\frac{1}{2}\sqrt{5}$, and $\frac{1}{2}+\frac{1}{2}\sqrt{5}$.

32. The product of two numbers is 10, and the product of their sum by the sum of their squares is 203; required the numbers found by a quadratic equation.

Ans. 5 and 2.

33. There are three numbers in geometrical progression, the difference of whose differences is 6, and the sum of the numbers 42; what are the numbers?

Ans. 24, 12, and 6.

34. There are three numbers in harmonical proportions, the difference of whose differences is 1, and the product of the extremes 18; what are the numbers?

Ans. 3, 4, and 6.

35. Required three equi-different numbers, such that if the first be increased by 1, the second by 2, and the third by the first, the sums may constitute an harmonical progression; but if 3 be added to the second, the sum may be a mean proportional between the sum of the numbers and the first diminished by $\frac{1}{2}$. Ans. 5, 6, and 7.

36. Required two such squares, that their difference shall be to the square root of the less as 3 to 7, and the square roots of the numbers to each other as 5 to 2.

Ans. $(\frac{10}{49})^2$, and $(\frac{4}{49})^2$.

37. Given the sum of the cubes of two numbers $=35$, and the sum of their 9th powers $=20195$; what are the numbers? Ans. 3 and 2

38. Given $9\frac{x^3}{y^3}+36\frac{x}{y}=85$, $\frac{3xy^3}{5}+\frac{51xy^2}{10}=\frac{102x}{5}+16$, required the values of x and y. Ans. $x=3\frac{1}{2}$, $y=2$.

39. What two numbers are those whose difference is 4, and their product multiplied by the sum of their squares 480? Ans. 6 and 2.

40. Given $x^2y\sqrt{xy}=a$, $xz^2\sqrt{xz}=b$, $y^2z\sqrt{zy}=c$, to find the values of x, y, and z.

Ans. $x=(\frac{a^4p^9}{c^6})^{\frac{1}{19}}$, $y=(\frac{c^4p^9}{b^6})^{\frac{1}{19}}$

$z=(\frac{b^4p^9}{a^6})^{\frac{1}{19}}$, p being $=\sqrt[4]{abc}$.

41. Required the values of x, y, and z, from the equations $yz=a$, $xz=b$, $xy=c$.

Ans. $x=\sqrt{\frac{bc}{a}}$, $y=\sqrt{\frac{ac}{b}}$, $z=\sqrt{\frac{ab}{c}}$.

42. There are three numbers in harmonical proportion, the sum of the first and third is 54, and their continued product 15552; what are the numbers?

Ans. 18, 24, and 36.

43. In how many different ways is it possible to pay £1000, without using any other coins than crowns, guineas, and moidores; a crown being 5*s*., a guinea 21*s*., and a moidore 27*s*.? Ans. 70734.

44. Given $x^2+y^2+z^2=266.5$, $x^2+y+z^2=176.5$, $(x+y+z)y=280$. Required the values of x y, and z. Ans. $x=10.5$, $y=10$, $z=7.5$.

45. Required two cube numbers, such that the first multiplied into the product of their roots, shall be equal to the second; but the second multiplied into the product of their roots, shall be equal to 64 times the first.

Ans. 8 and 64.

46. What number is that which being added to, and subtracted from 36, the sum of the cube roots shall be 6?

Ans. 28.

47. Given $x^2+y^2-x-y=249740$, $xy+x+y=8516$, to determine the values of x and y.

Result, $x=500$, $y=16$.

48. Given $x^4+y^4+x^2+y^2=238632-2x^2y^2$,
$y^4+z^4+z^2+y^2+2y^2z^2=1640$,
$x^2+y^2+z^2=\sqrt{(275100-x^2-y^2-z^2)}$
to find the values of x, y, and z.

Ans. $x=22$, $y=2$, $z=6$.

49. The sum of five numbers in geometrical progression is 242, and the fourth difference is 32, required the numbers. Ans. 2, 6, 18, 54, and 162.

50. Given $x+y+z+v+w=12.15=a$,
$x+y+z+vw=9.15=b$,
$x+y+zvw=5.68=c$,
$x+yzvw=5.09=d$,
$xyzvw=.45=e$. Required the values of x, y, z, v, and w.

Ans. $x=\dfrac{d\pm\sqrt{(d^2-4e)}}{2}=5$, or .09.

$$y=\frac{c-x\pm\sqrt{\{(c-x)^2-4(d-x)\}}}{2}$$

$$z=\frac{b-x-y\pm\sqrt{\{(b-x-y)^2-4(c-x-y\}}}{2}$$

$$v=\frac{a-x-y-z\pm\sqrt{\{(a-x-y-z)^2-4(b-x-y-z)\}}}{2}$$

$$w=a-x-y-z-v.$$

From the ambiguity of the signs, it is manifest that the number of numerical values of the unknown quantities go on increasing from x to v.

THE END.

ROBERT E. PETERSON,

BOOKSELLER AND PUBLISHER,

N. W. CORNER OF FIFTH AND ARCH STREETS,

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Invites attention to the following valuable works, just published.

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The attention of Teachers is invited to the following opinions of the work:

From Professor W. H. Allen, President of Girard College for Orphans, Philadelphia.

GIRARD COLLEGE, May 6, 1851.

ROBERT E. PETERSON.—Dear Sir—I beg leave to tender my thanks for your courtesy in sending me a copy of "Familiar Science." I have read parts of each division of the work, and have been pleased with the precision of the questions, and the accuracy of the answers. The book is not merely a volume of *familiar knowledge*, but a volume in which much rare and profound knowledge is *made familiar* to the common mind and applied to common things. I consider the book a valuable contribution to our means of instruction in schools, and hope to see it generally introduced and used by teachers. Fathers of families also, who are now frequently puzzled by the questions of the young philosophers of their households, will do well to procure a copy and avoid saying so often, "*I do not know.*" I remain truly yours, etc.,

WM. H. ALLEN.

From the Rev. Lyman Coleman, D. D., Principal of the Presbyterian Institute, Philadelphia.

"Familiar Science" embodies a vast amount of facts and principles, relating to the several branches of natural science, judiciously selected and arranged, and very useful to awaken inquiry in the young, and form a taste for such studies. For this purpose it is used as a text book in our school.

LYMAN COLEMAN, Prin.

PHILADELPHIA, May 15, 1851.

From the Right Rev. Bishop Potter.

BROAD STREET, June 2, 1851.

DEAR SIR—Absence from the city and urgent engagements have prevented me from acknowledging, as promptly as I wished, your note of last month. The same causes have prevented my giving to your work, entitled "Familiar Science," the thorough examination to which its apparent merit richly entitles it. So far as I have been able to look through it, it fully justifies your selection of it and the laborious revision which you have bestowed upon it. It contains a vast amount of useful information on subjects which force themselves upon the attention of both old and young, and it is likely to cultivate in those who read it habits of inquiry and reflection. The mechanical execution is entitled to unqualified praise. Very respectfully yours,

ALONZO POTTER.

MR. R. E. PETERSON.—Dear Sir—I am much pleased with the volume you left me for perusal, entitled "Familiar Science, or The Scientific Explanation of Common Things." It treats, in a familiar and interesting manner, of a large variety of subjects, which all children should understand, but many of which are indifferently understood by persons of mature age. The book cannot fail to interest the inquisitive pupil, and become a valuable auxiliary in the diffusion of *practical* science, both in schools and families.

Yours respectfully,

J. SIMMONS,

Principal of "The Locust Street Institute for Young Ladies."

PHILADELPHIA, May 17, 1851.

From N. C. Brooks, A. M., Principal of the Baltimore Female College. Late Principal of the Baltimore High School.

BALTIMORE, May 28, 1851.

Having examined the American edition of Dr. Brewer's Familiar Science, with additions by R. E. Peterson, I take pleasure in commending it to teachers and parents, and all others interested in the cause of education, as a most excellent work. It is worthy of very extensive circulation. I shall most probably have it used as a text book in our Institution.

N. C. BROOKS

From the Hon. Joel Jones, Ex-president of Girard College for Orphans Philadelphia

I take great pleasure in concurring in the foregoing recommendation. JOEL JONES.

Having carefully examined the book entitled "Familiar Science," I accord to it my decided approbation.

As a class book, in the hands of a judicious teacher, it cannot fail to strengthen the intellect and enlarge the useful knowledge of all young persons that may study it. So much am I impressed with its value, that I think it must speedily find its way into all good schools.

EDMUND SMITH, A. M.
"Franklin Hall," Newton University, Baltimore.
JUNE 2, 1851.

BURLINGTON, May 24, 1851.

ROBERT E. PETERSON, ESQ.—Dear Sir—It gives me great pleasure to acknowledge the receipt of a late reprint of yours entitled "Familiar Science." It affords another of the many examples, which are constantly being furnished nowadays, of the increased facilities of instruction. Such books as these implant in the youthful mind a desire and determination to inquire more extensively into science, and thus no one can calculate their value, for we are daily called upon to acknowledge the immense advantages which an insight into chemical action brings to bear upon mechanical, as well as agricultural fields of labor. I can well imagine the pleasure which a perusal of this little book of yours will afford to those, who first glean from its pages a knowledge of the subjects on which it treats, from a recollection of the feelings with which I used to read a much more unpretending volume of the same nature, now out of print, entitled "Common Things." Assuring you that it will give me great pleasure to recommend your book as an elementary one in the Institution with which I am connected,

I have the honor to subscribe myself
Very respectfully yours, GEORGE H. DOANE,
Instructor in Chemistry, and Lecturer on Physiology at Burlington College.

From Miss Phelps, Philadelphia.

Your excellent book, "Familiar Science," has been received, and, after having carefully examined it, I am prepared to concur entirely with the numerous testimonials you have already received of its merit as a text book for schools or families. It is eminently a book for the times and for the people, as a vast amount of general information on useful and practical subjects may be acquired without exploring multiplied abstruse matter for learning, comparatively, a few facts. I predict for it a ready and rapid sale, and con-

gratulate educators in having so able an auxiliary in the business of instruction. Thank you for the presentation of a volume, and

Am, very truly, H. M. PHELPS,
Prin. St. Mary's School.

PHILADELPHIA, May 13, 1851.

MR. ROBERT E. PETERSON.—Dear Sir—Many thanks to you for your present of "Familiar Science;" I have given it a careful perusal, and believe it to be better calculated to foster in the mind of youth a spirit of philosophic and scientific inquiry than any work I have ever seen. The questions are precise and the answers accurate; and the common things of every day life receive a profound and luminous explanation. The work is admirably arranged, and the index remarkably complete. It should be in all the schools in the country. I shall introduce it into mine at the next term.

With great respect, your friend,
D. R. ASHTON,
Female Institute, Philadelphia.

WECCACOE BOY'S GRAMMAR SCHOOL, May 15, 1851.

SIR—From an examination of the book entitled "Familiar Science," which you were pleased to send me, I feel convinced that it would be a valuable addition to the list of text-books now in use in our schools.

The importance of the subjects, and the familiar manner of treating of them, should make it a popular family, as well as school book.

JOHN JOYCE, Prin.

MR. ROBT. E. PETERSON.

MAY 16, 1851.

DEAR SIR—The pages of "Familiar Science" are its best recommendation. The common phenomena of life are treated of in a simple and intelligible manner, which renders it both pleasing and instructive. In the family circle, as a text book, it will form the basis of an hour's interesting conversation, and in the hands of the pupil it will be a valuable aid in the acquisition of useful knowledge.

Respectfully,
WM. S. CLEAVENGER,
Prin. Boys' Grammar School, Locust St.

TO ROBERT E. PETERSON, ESQ.

PHILADELPHIA, May 17, 1851.

DEAR SIR—I am highly pleased with the treatise on "Familiar Science, or the Scientific Explanation of Common Things," of which you sent me a copy for examination. I have examined the work and find it contains, within a comparatively small compass, a great amount of useful information. It is a work that should be in

the hands of every person, both young and old, who has any desire to become acquainted with the common phenomena of life.

Respectfully yours, etc.

SAMUEL F. WATSON
Prin. of Mount Vernon Boys' Grammar School.

ROBERT E. PETERSON, ESQ.

ROBERT E. PETERSON, ESQ.—Dear Sir—I have been much gratified by an examination of your book entitled "Familiar Science," and believe its introduction into schools and families would do more in imparting a general knowledge of scientific principles than any treatise ever used.

The cause of every day phenomena, such as evaporation, condensation, the formation of clouds, rain, dew, etc., are so familiarly explained, that all classes of persons may readily comprehend them, and I believe the book has only to be known to be appreciated by teachers. I am truly yours,

WM. ROBERTS,
Prin. of Ringgold Grammar School.

PHILADELPHIA, May 19, 1851.

PHILADELPHIA, May 13, 1851.

ROBERT E. PETERSON, ESQ.—Dear Sir—I am highly pleased with your "Familiar Science." As a teacher, I am constantly called upon to answer questions upon almost every conceivable subject, and children wish, and should have correct and intelligent answers. Your two thousand questions and answers, which I consider most admirable, will much facilitate the labors of parents and teachers.

Yours truly,

J. H. BROWN, A. M.
Prin. of Zane St. Boys' Grammar School

From Professor James C. Booth, A. M., M. A., P. S. Author of the Encyclopedia of Chemistry; Melter and Refiner in the U. S. Mint; Professor of Applied Chemistry in the Franklin Institute.

ANALYTICAL LABORATORY, COLLEGE AVENUE,
PHILADELPHIA, July 3, 1851.

DEAR SIR—I have examined "Familiar Science" with some care, and must express a hearty approval of the manner in which the most "common things" of life are familiarly and clearly explained, without sacrificing the correctness of science. Embracing such questions as are usually put by the developing mind of children, with clear and precise answers, it will relieve parents and teachers of the unhappy necessity of crushing youthful inquiry, while it will tend to nourish a spirit of reflection and investigation in young and old. I commend it as a valuable catechism for schools, and for amusement and instruction at the fire-side.

Respectfully yours, JAS. C. BOOTH.

MR. R. E. PETERSON, PHILA.

From Mrs. Phelps, favorably known as the author of various popular school books on Chemistry, Botany, Philosophy, Geology, etc. etc.

MR. PETERSON.—Sir—In thanking you for the book called "Familiar Science," permit me to express the satisfaction with which it is received by myself and the teachers and pupils of this Institution; having labored for many years to render Science Familiar, I am glad to meet with this unpretending assistant. It is recommended by the fact that if adopted in a school, it will crowd out no other book, since it is *sui generis* and may be studied as an amusement.

With respect, yours, etc.

ALMIRA LINCOLN PHELPS,

Patapsco Institute, near Ellicott's Mills, Md.

JUNE 30, 1851.

From the Rev. Father Sourin.

PHILADELPHIA, July 4, 1851.
ST. JOHN'S CATHEDRAL.

DEAR SIR—I comply most willingly with your request, though after the high and well-deserved praise which your valuable work has already received, further commendation would appear unnecessary. An examination of its contents will satisfy every intelligent reader that it is a treasure of various, useful, and entertaining knowledge, well calculated to promote one of the chief purposes of true education, viz. to make us more familar with those works of God (so often full of singular beauty and power when, seemingly, most common) in the midst of which "we live and move, and have our being." It will afford me much pleasure to aid in its circulation.

Respectfully, your obed't serv.,

EDW. J. SOURIN.

R. E. PETERSON, ESQ.

From T. S. Arthur, Editor of the Home Gazette.

"Familiar Science, or the Scientific Explanation of Common Things," edited and published by Robert E. Peterson, of this city, is one of the most generally useful books that has lately been printed. This work, or a portion of it, came first from the pen of the Rev. Doctor Brewer, of Trinity Hall, Cambridge, but, in the form it first appeared from the English press, it was not only unsuited to the American pupil, but very deficient in arrangement. These defects, the editor has sought to remedy. To give not only to the parent a ready means of answering inquiries, but to provide a good book for schools, is the object of this volume. About two thousand questions, on all subjects of general information, are answered in language so plain that all may understand it. We had marked a number of these for publication, but must defer their insertion to another number.

The book is very strongly recommended, and not without good

reason, by competent judges of its merits, and we add our own testimony in its favor.

From the Episcopal Recorder.

We are here furnished with one of those time and labor-saving volumes which, by judicious arrangement, simple illustration, and accurate definition, greatly facilitate the important effort to impart a familiar knowledge of the facts and principles of natural science. To all interested in the education of youth, it will be found a valuable auxiliary, while its study, in families and schools, cannot fail to enlarge the domain of Science, by fixing in the mind the elementary principles of the every day phenomena of life.

From the Saturday Gazette.

One of the most valuable "family library" books we have ever met. The title explains the object of the work, which embraces a wonderful variety of topics cleverly treated. Twenty-five thousand copies of the English edition were sold in two years, and the present publisher, also the editor, has made it worthy of as large a circulation, by the clear type and paper and the durable binding. We commend it to all parents or teachers of the young.

From Fitzgerald's City Item.

Occasionally a book is published that fills a void that seems to have been left for it—that exactly furnishes what has been often wished for. Such a work is "Familiar Science." It gives concisely, yet clearly, about two thousand explanations of the scientific principles involved in the most frequently recurring operations of life. It is an invaluable book for all whose relation with the young, whether as parent or teacher, demand a facility in explaining what so often attracts the curiosity of childhood. To the general reader, too, it opens an interesting field for observation; and even to the scholar accustomed to philosophical investigation, it may present familiar applications of what are often regarded as abstruse and unpractical researches.

From the Saturday Evening Post.

Here is a book which might find an appropriate place in every school, and in every house where there are young inquiring minds, athirst for knowledge. It contains nearly two thousand questions, with scientific answers, simply arranged and clearly expressed, conveying, in a small compass, a great amount of information respecting the every-day wonders which puzzle the restless brains of children, and sometimes their elders, too. The publisher and editor merits, and will no doubt receive, the thanks of all parents and

teachers who have experienced (and what instructor of youth has not?) the difficulty of meeting, with satisfactory explanations, the innumerable questions of those under their charge. The book needs only to be known, to be pronounced by a discerning public a rich fund of useful entertainment for the household circle, as well as a valuable aid to instruction in schools. We are not surprised at the statement in the preface, that "Twenty-five thousand copies of the English edition of this work were sold in London in less than two years;" neither shall we be astonished if double that number are sold in the United States in even a shorter space of time.

From the Banner of the Cross.

It is an exceedingly interesting, as well as instructive book for children. It teaches them the reasons of many things, which though common, are yet mysterious; and it also gives many practical hints upon matters which relate to the experience of every-day life. We can heartily unite with the many strong recommendations from those whose opinion in such matters are especially valuable, in their favorable judgment of this work, as a most excellent book for school or family use.

From the Christian Observer.

This is an excellent and valuable book for the young in families and schools. It opens to the reader a rich repository of science respecting the most common things—facts and lucid explanations of them which cannot fail to interest and instruct those for whom it is intended. We commend it to the public as a valuable book for the home library and the school.

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SCHOOL TEACHERS AND HEADS OF FAMILIES.

FAMILIAR SCIENCE,

OR

SCIENTIFIC EXPLANATIONS [illegible]

BY REV. [illegible]

EDITED BY R. E. PETERSON,

Member of the Academy of Natural Sciences, Philadelphia.

CERTIFICATES.

At a meeting of the Controllers of Public Schools, First District of Pennsylvania, held at the Controller's Office, Tuesday, Sept. 9, 1851, the following [illegible]:

Resolved, That Peterson's "Familiar Science" be introduced into the Grammar Schools of this District.

[illegible] J. HEMPHILL, *Sec'y.*

At a meeting of the Board of Education of the Brooklyn Public Schools, held Dec. 2, 1851, the following [illegible] was adopted:

Resolved, That "Peterson's Familiar [illegible]" be adopted as a text book, [illegible] in the Public Schools.

[illegible] DILLINGHAM,

Attest: *Chairman of [illegible] School Books.*

S. L. HIGGINS, *Sec'y.*

From the Rev. [illegible] President [illegible] College.

MR. R. E. PETERSON: [illegible] to the city, after long absence [illegible] "Familiar Science," [illegible] I have read [illegible] excellent [illegible] and [illegible] by the ordinary [illegible] accordance with [illegible] Christianity [illegible] Church. This Christian aspect of the [illegible] book [illegible] caution, but is clear and candid, a [illegible] may be seen in the questions and answers on [illegible] miracles of Gideon, as recorded in [illegible] chapter of the Book of Judges. I consider the volume [illegible] contribution to Christian scientific literature, and worthy [illegible] every family and in every academy or school.

[illegible] respectfully, J. P. DURBIN.